1. Suppose that α and β are reals and $\sum a_n$ and $\sum b_n$ are two convergent series.
   (i) Show that the series $\sum(\alpha a_n + \beta b_n)$ is convergent.
   (ii) Find an example such that $\sum a_n b_n$ diverges.
   (iii) Owing to (ii) discover a natural series defined in terms of $\sum a_n$ and $\sum b_n$ which converges and converges to $(\sum a_n)(\sum b_n)$.

2. Discuss the convergence (or divergence) of the following series.
   (a) $\sum \frac{1}{(n+1)(n+2)}$
   (b) $\sum \frac{n}{2^n}$
   (c) $\sum \frac{(n!)^2}{(2n)!}$
   (d) $\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \cdots$

   {Use a calculator/computer to evaluate first 'few' terms of the sequence of partial sums, to get a grip.}

3. For real $|r| < 1$ and natural k, investigate the convergence or divergence of

$$\sum n^k r^n, \quad \sum k^n r^n, \quad \sum n! r^n, \quad \sum \frac{n^k}{k^n}, \quad \sum \frac{n^k}{n!}, \quad \sum \frac{k^n}{n!}.$$

4. For $i = 1, 2, 3, 4, 5, \ldots,$ let
   (a) (a_n^1) denote the sequence $+1, +1, -1, +1, +1, -1, \ldots$
   (b) (a_n^2) be the sequence $+1, -1, +1, +1, -1, +1, \ldots$
   (c) (a_n^3) be the sequence $-1, +1, +1, -1, +1, +1, \ldots$
   (d) (a_n^4) be the sequence $+1, +1, +1, -1, +1, +1, +1, -1, \ldots$

   Investigate the convergence or divergence of each of the series $\sum \frac{a_n^i}{n}$.

5. (a) For every natural n, let real $a_n > 0$. Assume $\sum a_n$ is convergent. Prove that $\sum a_n^2$ is convergent.
   (b) Give an example of a series such that $\sum a_n$ is convergent while $\sum a_n^2$ is divergent.

6. (a) If $\sum a_n$ is absolutely convergent and (b_n) is a bounded sequence, show that $\sum a_n b_n$ is absolutely convergent.
   (b) Give an example for an absolutely convergent $\sum a_n$ and an unbounded sequence (b_n) such that $\sum a_n b_n$ diverges.
   (c) Give an example for a conditionally convergent $\sum a_n$ and a bounded sequence (b_n) such that $\sum a_n b_n$ diverges.

7. If (a_n) is a sequence and if $\lim(n^2 a_n)$ exists, show that $\sum a_n$ is absolutely convergent.

8. Problem 8(f) below.

9. Does the series

$$\frac{1}{1} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \frac{1}{9} - \frac{1}{10^2} + \cdots,$$

   converge or diverge? Investigate with justifications. (Formal definition: The series is $\sum a_n$, where the sequence satisfies $a_{2n-1} = \frac{1}{2n-1}$ and $a_{2n} = -\frac{1}{(2n)^2}$)

10. Show that there exists a rearrangment of the alternating harmonic series whose sequence of partial sums has neither an upper nor a lower bound.

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11. Given any sequence (a_n), define its positive part as the sequence $a_n^+ = \max(a_n, 0)$ and its negative part as $a_n^- = \min(a_n, 0)$. Denote $\sum a_n^+$, $\sum a_n^-$, $\sum a_n$ and $\sum |a_n|$ by P, N, C and A respectively. Show the following.
    (a) If both P and N converge, then both C and A converge. Find their limits.
    (b) If one of P or N converges while the other diverges, then both C and A diverge.
    (c) If both P and N diverge then at most one of C or A converges and there are examples where both C and A diverge.

(d) If both C and A converge, then both P and N converge.

(e) Is there an example where A converges while C diverges?

(f) If C converges while A diverges (i.e., the series $\sum a_n$ is conditionally convergent), then both P and N diverge.

(g) If both C and A diverge, then at most one of P or N converges and there are examples where both P or N diverge.

12. If α, β are reals of the same sign and p is any natural prove

$$\sum \frac{1}{\alpha+n\beta} \frac{1}{\alpha+(n+1)\beta} \cdots \frac{1}{\alpha+(n+p)\beta} = \frac{1}{p\beta} \cdot \frac{1}{\alpha+\beta} \frac{1}{\alpha+2\beta} \cdots \frac{1}{\alpha+p\beta}$$

13. Suppose $\sum a_n$ is conditionally convergent. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijection. Define $W = \{|\varphi(n) - n| \,|n \in \mathbb{N}\}$. Investigate the conjecture: $\sum a_{\varphi(n)} = \sum a_n$ if and only if W is bounded.

14. {Problem 1(iii) was about multiplying series. This one is about dividing two series.} Let (s_n) and (t_n) be the sequence of partial sums of the series $\sum a_n$ and $\sum b_n$. Assume that $t_n \neq 0$ for each natural n. Define a sequence (q_n) via $q_1 := \frac{s_1}{t_1}$ and $q_{n+1} := \frac{a_{n+1}t_n - b_{n+1}s_n}{t_n t_{n+1}}$. Prove that $\sum q_n$ converges to $\sum a_n / \sum b_n$.

15. For a sequence (a_n) define its *signed variation* to be $\sum(a_{n+1} - a_n)$ and its (unsigned or total) *variation* to be $\sum |a_{n+1} - a_n|$. Say that (a_n) has *bounded variation* if its variation converges.

(a) Prove that a sequence converges if and only if its signed variation converges.

(b) Prove that a sequence of bounded variation converges.

(c) Find examples of a convergent sequences which are and are not of bounded variation.

(d) Prove that a monotonic sequence is of bounded variation if and only if it is bounded.

(e) Prove that a sum, difference, product and constant multiple of sequences of bounded variation are of bounded variation. How about a quotient?

(f) Every sequence of bounded variation can be demonstrated to be a difference of two monotonic bounded sequences.