MA 201 (Part II), July-November, 2022 session
Partial Differential Equations
Tutorial Problem Sheet 10
Date of Discussion: November 18, 2022

Laplace Transform: Properties and applications

Fourier Integral and Transform: Properties and applications

Lectures 16-19

1. Find the Laplace transforms of

$$\text{(i) } te^{3t}\cos 4t, \quad \text{(ii) } t\int_0^t e^{-3t}\sin 2t\, dt, \quad \text{(iii) } \int_0^t \frac{e^{-3t}\sin 2t}{t}\, dt.$$

2. Find the Laplace transform of the following functions involving unit step function: (i) $2H(\sin \pi t) - 1$, (ii) $H(t^3 - 6t^2 + 11t - 6)$.

3. Find the inverse Laplace transforms: (i) $\dfrac{2s+3}{s^2+4s+6}$, (ii) $\dfrac{2s^2-3s+5}{s^2(s^2+1)}$.

4. Find the inverse Laplace transforms of the following by the theory of residues: (i) $\dfrac{1}{(s+1)(s-2)^2}$, (ii) $\dfrac{s+2}{(s+1)(s^2+4)}$.

5. Solve the following Initial Value Problems: (i) $\ddot{y} + 2\dot{y} + 5y = e^{-t}\sin t,\ y(0) = 0,\ \dot{y}(0) = 1,$ (ii) $t\ddot{y} + 2\dot{y} + ty = 0,\ y(0) = 1.$

6. Solve the following IBVP for one dimensional heat conduction equation for a rod of unit length and with unit diffusivity:

$$\begin{aligned} &U_t = U_{xx},\ 0 < x < 1, t > 0,\\ &U(x,0) = 3\sin(2\pi x),\ 0 < x < 1,\\ &U(0,t) = 0 = U(1,t),\ t > 0. \end{aligned}$$

7. Find the Fourier integral representation of the following non-periodic function:

$$f(t) = \begin{cases} \sin t, & t^2 \le \pi^2,\\ 0, & t^2 > \pi^2. \end{cases}$$

8. Find the Fourier integral representation of the following non-periodic function

$$f(t) = \begin{cases} 0, & -\infty < t < -1,\\ -1, & -1 < t < 0,\\ 1, & 0 < t < 1,\\ 0, & 1 < t < \infty. \end{cases}$$

9. Express

$$f(x) = \begin{cases} 1, & 0 \le x \le \pi, \\ 0, & x > \pi, \end{cases}$$

as a Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1-\cos(\pi\sigma)}{\sigma} \sin(\pi\sigma) d\sigma.$$

10. If $U(x,t)$ is the temperature at time t and α the thermal diffusivity of a semi-infinite metal bar, find the temperature distribution in the bar at any point at any subsequent time by solving the following initial boundary value problem:

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \quad x > 0, \ t > 0,$$
$$\frac{\partial U}{\partial x}(0,t) = 0, \quad U(x,0) = f(x).$$

Practice Problems

11. Find the Fourier integral representation of the following rectangular pulse:

$$f(t) = \begin{cases} 1 & t^2 \le b^2, \\ 0 & t^2 > b^2. \end{cases}$$

12. Consider the wave propagation in a semi-infinite string, fastened at the end $x = 0$ with initial displacement $f(x)$ and initial velocity $g(x)$. Find the vibration $U(x,t)$ at any point at any subsequent time, by using the appropriate Fourier transform.

13. Find the Laplace transform of the following function $F(t)$:

$$F(t) = \begin{cases} t, & 0 \le t < 1, \\ 2-t, & 1 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

14. Using convolution theorem, find the following:
(i) $\mathcal{L}^{-1}\left\{\frac{s^2+4s+4}{(s^2+4s+13)^2}\right\}$, (ii) $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2(s+2)^2}\right\}$.