MA 201 (Part II), July-November, 2022 session
Partial Differential Equations
Tutorial Problem Sheet 4, Date of Discussion: November 11, 2022

Separation of variables: one-dimensional wave equation

One-dimensional diffusion equation, two-dimensional steady-state equation

Lectures 11-15

1. A string is fixed at $x = 0$ and $x = L$ and lies initially along the x-axis. If it is set in motion by giving all points $0 < x < L$ a constant transverse velocity $\frac{\partial u}{\partial t} = u_0$ at $t = 0$, then find the subsequent motion of the string.

2. A guitar string of length $L = 1$, is pulled in the middle so that it reaches a height h. Assuming the initial position of the string as

$$u(x,0) = \begin{cases} 2hx, & 0 < x < 1/2, \\ 2h(1-x), & 1/2 \le x < 1, \end{cases}$$

what is the subsequent motion of the string if it is suddenly released?

3. A metal bar of length 100 metre has its ends $x = 0$ and $x = 100$ kept at 0 degree Celsius. Initially half of the bar is at 60 degree C while the other half is at 40 degree C. Assuming a thermal diffusivity of 0.16 cgs units and that the surface of the bar is insulated, find the temperature everywhere in the bar at time t.

4. Solve the following IBVP with non-homogeneous BCs:

$$\begin{aligned} &u_t = u_{xx},\ 0 < x < 1, \\ &u(0,t) = 0, \\ &u(1,t) = 1, \\ &u(x,0) = x^2. \end{aligned}$$

5. Using Duhamel's principle for an infinite string problem, solve

$$\begin{aligned} &u_{tt} - u_{xx} = x - t,\ -\infty < x < \infty, \\ &u(x,0) = 0, \quad u_t(x,0) = 0. \end{aligned}$$

6. Find a solution $u(x,y)$ of the following steady-state heat conduction problem

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \quad 0 < x < \pi,\ 0 < y < 1, \\ &u(x,0) = 0, \quad 0 < x < \pi, \ u(x,1) = \begin{cases} x, & 0 < x < \pi/2, \\ \pi - x, & \pi/2 < x < \pi, \end{cases}, \\ &u(0,y) = u(\pi,y) = 0, 0 < y < 1. \end{aligned}$$

7. Consider transient heat conduction in a circular region of radius a. Considering that heat conduction takes place only radially, find the solution of the transient heat conduction in the circular disk at any point r at any time $t > 0$
(i) When the boundary is kept at zero degree C and the initial temperature distribution is given by $u(r,0) = 100$ (in C),
(ii) When the boundary is kept at zero degree C and the initial temperature distribution is given by $u(r,0) = r$ (in C).

8. Solve the following boundary value problem inside a circular disk:

$$\nabla^2 u = 0, r < a,\ 0 \le \theta < 2\pi,$$
$$u(a, \theta) = 4 + 3\sin\theta, 0 \le \theta < 2\pi.$$

Practice Problems

The following problems will not be discussed in the tutorial class.

9. Change the initial conditions in Problem 1 above to

$$u(x,0) = \begin{cases} x, & 0 < x < L/2, \\ (L-x), & L/2 \le x < L. \end{cases}$$

and the string is released from rest at this position. Hence find the displacement at any subsequent time.

10. Find a solution of the wave equation $u_{tt} = u_{xx};\ \ 0 < x < \pi,\ t > 0$, subject to the following boundary and initial conditions:

$$u(0,t) = u(\pi,t) = 0, t > 0;\ u(x,0) = \pi x - x^2,\ u_t(x,0) = 0,\ 0 < x < \pi$$

11. A metal bar of length L is initially at a temperature given by

$$f(x) = \begin{cases} x, & 0 < x < L/2, \\ L-x, & L/2 \le x < L, \end{cases}$$

and its ends are kept at zero degree C. Find the temperature distribution of the bar at any time.

12. Find the temperature $u(x,t)$ in a bar of length L which is perfectly insulated, also at the ends $x = 0$ and $x = L$ such that $u_x(0,t) = u_x(L,t) = 0$, and the initial temperature in the bar is $u(x,0) = f(x)$. Also, find the temperature in the bar, given $L = \pi, \alpha = 1$, for
(i) $f(x) = 1$, (ii) $f(x) = x^2$.

13. Using Duhamel's Principle, solve the following:

$$\begin{aligned} u_{tt} &= u_{xx} + x\sin t,\ \ 0 < x < 1,\ \ t > 0, \\ u(x,0) &= x(1-x),\ \ u_t(x,0) = 0,\ 0 \le x \le 1, \\ u(0,t) &= u(1,t) = 0,\ \ t > 0. \end{aligned}$$

14. Using Duhamel's Principle, solve the following:

$$\begin{aligned} u_t &= u_{xx} + t[2x + \sin 2\pi x],\ \ 0 < x < 1,\ \ t > 0, \\ u(x,0) &= 1 + \sin(\pi x) - x,\ 0 \le x \le 1, \\ u(0,t) &= 1,\ \ u(1,t) = t^2,\ \ t > 0. \end{aligned}$$

15. Find a solution $u(r,\theta)$ of the boundary value problem

$$\begin{aligned} r^2 u_{rr} + r u_r + u_{\theta\theta} &= 0,\quad 0 < r < 1,\ 0 < \theta < \pi, \\ u(0,\theta) &= 0,\ \ u(1,\theta) = \theta(\pi - \theta),\quad 0 < \theta < \pi, \\ u(r,0) &= u(r,\pi) = 0,\quad 0 < r < 1. \end{aligned}$$