MA 201 (Part II), July-November, 2022 session
Partial Differential Equations
Tutorial Problem Sheet - 3, Date of Discussion: October 28, 2022

Fourier series

Lectures 9 and 10

1. Find the Fourier series of the following functions:

$$(a) \qquad f(x) = \begin{cases} -x, & -\pi \leq x \leq 0, \\ x, & 0 \leq x \leq \pi. \end{cases}$$
$$(b) \qquad f(x) = |\sin x|, \quad -\pi < x < \pi.$$

2. Find the Fourier series expansion for the function $f(x)$ as given:
(a) $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$
(b) $f(x) = \begin{cases} -\pi/2, & -\pi < x < 0, \\ \pi/2, & 0 < x < \pi. \end{cases}$
(c) $f(x)$ is given by the line joining $(-\pi, 0)$ and $(0, 2)$ in $(-\pi, 0)$ and given by the line $f(x) = 2$ in $(0, \pi)$.

3. For the following functions, find the Fourier cosine series and the Fourier sine series on the interval $0 < x < \pi$:
(a) $f(x) = 1$, (b) $f(x) = \pi - x$, (c) $f(x) = x^2$.

4. Given the Fourier series for the function $f(x) = x^4, \quad -\pi < x < \pi$, as

$$x^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^4}(\pi^2 n^2 - 6)\cos nx$$

find the Fourier series for $f(x) = x^5, \quad -\pi < x < \pi$.

5. Deduce the Fourier series for the function $f(x) = e^{ax}, \ -\pi < x < \pi, \ a$ a real number. Hence find the values of the four series:

$$\text{(a)} \ \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}, \quad \text{(b)} \ \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}, \quad \text{(c)} \ \sum_{n=1}^{\infty} \frac{1}{a^2+n^2}, \quad \text{(d)} \ \sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2}.$$

6. Consider $f(x) = \sqrt{1 - \cos x}, \ 0 < x < 2\pi$.
(a) Determine Fourier series expansion of f in $(0, 2\pi)$.
(b) Does the limit of Fourier series exist at $x = 0$?
(c) Use part (b) to find the series

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots.$$

7. Given the half-range sine series

$$x(\pi - x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}, \ 0 \leq x \leq \pi,$$

use Parseval's theorem to deduce the value of the series $\sum_{n=1}^{\infty} 1/(2n-1)^6$.