MA 201 (Part II), July-November, 2022 session
Partial Differential Equations
Problem Sheet - 2, Date of Discussion: October 21, 2022

Topics: 2nd order PDEs with constant coefficients, Classification of 2nd order PDEs,
Canonical forms, The wave equation: Infinite string problem (D'Alembert's solution)

Lectures 6−8

1. Find the general solution of

$$\begin{aligned}&\text{(i)}\ \ 3u_{xx} + 10u_{xy} + 3u_{yy} = 0,\\ &\text{(ii)}\ \ u_{xx} + 4u_{xy} + 4u_{yy} = 0,\\ &\text{(iii)}\ \ u_{xxx} - 2u_{xxy} - u_{xyy} + 2u_{yyy} = 0.\end{aligned}$$

2. Why is it so that only the principal part $Au_{xx} + Bu_{xy} + Cu_{yy}$ of the 2nd-order PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$ determines the nature of the PDE?

3. Classify the following second-order partial differential equations:
(i) $u_{xx} + 4u_{xy} + 4u_{yy} - 12u_y + 7u = x^2 + y^2$; (ii) $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x + y)$
(iii) $(x + 1)u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = 0$; (iv) $yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$.

4. Reduce the following equations to canonical form and hence solve them:
(i) $u_{xx} + 4u_{xy} + 3u_{yy} = 0$; (ii) $4u_{xx} - 12u_{xy} + 9u_{yy} = e^{3x+2y}$,
(iii) $u_{xx} + 2u_{xy} + u_{yy} = x^2 + 3\sin(x - 4y)$

5. Find D'Alembert's solution of one-dimensional wave equation with the following initial conditions:

$$\text{(i)}\ u(x, 0) = \sin x,\ \ u_t(x, 0) = 0, \qquad \text{(ii)}\ u(x, 0) = \sin x,\ \ u_t(x, 0) = \cos x.$$

6. A string stretching to infinity in both directions is given the initial displacement

$$\phi(x) = \frac{1}{1 + 4x^2}$$

and released from rest. Find its subsequent motion as a function of x and t.