MA 201 (Part II), July-November, 2022 session

Partial Differential Equations

Problem Sheet - 1, Date of Discussion: October 14, 2022

Derivation of PDEs, General integrals, Cauchy problems,

Integral surface through given curves, Orthogonal surfaces

Lectures 1−5

1. Find the partial differential equation arising from each of the following surfaces:
   (i) $u = f(x-y)$, (ii) $2u = (ax+y)^2 + b$, (iii) $f(x^2+y^2,\, x^2-u^2) = 0$.
   (x, y: independent variables, u: dependent variable, a, b: arbitrary constants, f: arbitrary function.)

2. Is it must that a well-posed initial value problem possesses a unique solution? Justify your answer.

3. What is a characteristic curve? How does a PDE with two independent variables become an ODE along the characteristic curve?

4. Find the general integral of the following partial differential equations:
   (i) $x^2p + y^2q + u^2 = 0$, (ii) $x^2(y-u)p + y^2(u-x)q = u^2(x-y)$.

5. Show that the integral surface of the equation $2y(u-3)p + (2x-u)q = y(2x-3)$ that passes through the circle $x^2+y^2 = 2x$, $u = 0$ is $x^2+y^2-u^2-2x+4u = 0$.

6. Find the solution of the following Cauchy problems:
   (i) $u_x + u_y = 2$, $u(x,0) = x^2$; (ii) $5u_x + 2u_y = 0$, $u(x,0) = \sin x$.

7. Show that the Cauchy problem $u_x + u_y = 1$, $u(x,x) = x$ has infinitely many solutions.

8. Consider the PDE $xu_x + yu_y = 4u$, where $x, y \in \mathbb{R}$. Find the characteristics curves for the equation and determine an explicit solution that satisfies $u = 1$ on the circle $x^2+y^2 = 1$.

9. Find a function $u(x,y)$ that solves the Cauchy problem

   $$x^2u_x + y^2u_y = u^2, \quad u(x,2x) = x^2, \quad x \in \mathbb{R}.$$

   Is the solution defined for all x and y?

10. Find the surface which is orthogonal to the one-parameter system

    $$u = cxy(x^2+y^2)$$

    and which passes through the hyperbola $x^2 - y^2 = a^2$, $u = 0$.