

MA 201 Complex Analysis  
Lecture 11: Applications of Cauchy's Integral  
Formula

**Cauchy's estimate:** Suppose that  $f$  is analytic on a simply connected domain  $D$  and  $\overline{B(z_0, R)} \subset D$  for some  $R > 0$ . If  $|f(z)| \leq M$  for all  $z \in C(z_0, R)$ , then for all  $n \geq 0$ ,

$$|f^n(z_0)| \leq \frac{n!M}{R^n},$$

where  $C(z_0, R) = \{z : |z - z_0| = R\}$ .

**Proof:** From Cauchy's integral formula and  $ML$  inequality we have

$$\begin{aligned} |f^n(z_0)| &= \left| \frac{n!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \\ &\leq \frac{n!}{2\pi} M \frac{1}{R^{n+1}} 2\pi R = \frac{n!M}{R^n}. \end{aligned}$$

# Liouville's Theorem

**Liouville's Theorem:** If  $f$  is analytic and bounded on the whole  $\mathbb{C}$  then  $f$  is a constant function.

**Proof:** By Cauchy's estimate for any  $z_0 \in \mathbb{C}$  we have,

$$|f'(z_0)| \leq \frac{M}{R}$$

for all  $R > 0$ . This implies that  $f'(z_0) = 0$ . Since  $z_0$  is arbitrary and hence  $f' \equiv 0$ . Therefore  $f$  is a constant function.

- $\sin z, \cos z, e^z$  etc. can not be bounded. If so then by Liouville's theorem they are constant.

# Liouville's Theorem

- Does there exist a non constant entire function  $f$  such that  $e^{f(z)}$  is bounded?
- Does there exist a non constant entire function  $f$  such that  $\operatorname{Re}(f)$  is bounded?
- Does there exist a non constant entire function  $f$  such that  $\operatorname{Im}(f)$  is bounded?
- Does there exist a non constant entire function  $f$  such that  $f(x)$  is bounded for all real  $x$ ?
- Does there exist a non constant entire function  $f$  such that  $|f(z)| > 1$  for all  $z \in \mathbb{C}$ ?

# Fundamental Theorem of Algebra

- **Fundamental Theorem of Algebra:** Every polynomial  $p(z)$  of degree  $n \geq 1$  has a root in  $\mathbb{C}$ .
- **Proof:** Suppose  $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$  is a polynomial with no root in  $\mathbb{C}$ . Then  $\frac{1}{P(z)}$  is an entire function.
- Since

$$\left| \frac{P(z)}{z^n} \right| = \left| 1 + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right| \rightarrow 1, \quad \text{as } |z| \rightarrow \infty,$$

- It follows that  $|p(z)| \rightarrow \infty$  and hence  $|1/p(z)| \rightarrow 0$  as  $|z| \rightarrow \infty$ .
- Consequently  $\frac{1}{p(z)}$  is a bounded function.
- Hence by Liouville's theorem  $\frac{1}{p(z)}$  is constant which is impossible.

# Morera's Theorem

**Morera's Theorem:** If  $f$  is continuous in a simply connected domain  $D$  and if

$$\int_C f(z)dz = 0$$

for every simple closed contour  $C$  in  $D$  then  $f$  is analytic.

**Proof:** Fix a point  $z_0 \in D$  and define

$$F(z) = \int_{z_0}^z f(w)dw.$$

Use the idea of proof of existence of antiderivative to show that  $F' = f$ . Now by Cauchy integral formula  $f$  is analytic.