MA 542 Differential Equations

Lecture 8

(January 20, 2022)

Second-order equations have immense significance from practical point of view.

- Many physical phenomena can be represented in terms of second-order differential equations
- Many aspects of mathematical physics contain second-order differential equations

Most general form:

$$y''(x) = f(x, y, y')$$

Specific form:

$$y'' + P(x)y' + Q(x)y = R(x).$$

Solutions depend on the coefficient functions:

- $P(x)$ and $Q(x)$ are constants.
- $P(x)$ and $Q(x)$ have a specific form in x (Cauchy-Euler form).
- $P(x)$ and $Q(x)$ have an arbitrary form in x with singularities.

For $P(x)$ and $Q(x)$ with regular singularities, power series solution is considered to be one of the most powerful methods.

Need for a power series solution:

Most of the specific functions encountered in elementary analysis belong to a class known as the elementary functions. Recall that an algebraic function is a polynomial, a rational function or more generally any function $y = f(x)$ that satisfies an equation of the form

$$p_n(x)y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0(x) = 0$$

where each p_i is a polynomial.

The elementary functions consist of the algebraic functions; the elementary transcendental (or non-algebraic) functions such as trigonometric, inverse trigonometric, exponential and logarithmic functions; and all other functions that can be constructed from these by adding, subtracting, multiplying, dividing or forming a function of a function.

Consider

For example, consider the following which looks complicated but is still an elementary function:

$$f(x) = \tan\left[\frac{xe^{1/x} + \tan^{-1}(1 + x^2)}{\sin x \cos 2x - \sqrt{\ln x}}\right]^{1/3}$$

Beyond the elementary functions lie the higher transcendental functions or special functions. Most of them arise as solutions of second-order linear ordinary differential equations. Many of these special functions find application in connection with partial differential equations in mathematical physics.

The study of special functions has been developed and strengthened by great mathematicians like Euler, Gauss, Abel, Jacobi, Hermite and many more.

Let us see how these functions come into picture. We know that the simple ODE $y'' + y = 0$ can be solved to get the familiar functions $y = \cos x$ and $y = \sin x$ as the solutions.

Now, consider the equation $xy'' + y' + xy = 0$ which cannot be solved in terms of elementary functions. In other words, we do not know any familiar procedure which will yield solutions for this equation. We have to look for some alternative method to solve such type of equations: solve it in terms of power series and we use these series to define new special functions.

Definition

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots , \tag{1}$$

where $a_0, a_1, a_2, \cdots, a_n, \cdots$ are constants, is called a power series in x (about the origin).

Definition

The series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \tag{2}$$

is a power series in $(x - x_0)$, which looks more general than equation (1). But equation (2) can be reduced to the form (1) by writing x for $(x - x_0)$ which is nothing but only a translation of the coordinate system.

Definition

The series (1) is said to converge at a point x if the limit $\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ exists and in this case the sum of the series is the value of this limit.

Radius of convergence

Obviously, the series (1) always converges at the point $x = 0$. With respect to the arrangement of their points of convergence, all power series in x fall into one of the following three main categories of the type:

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2!\, x^2 + 3!\, x^3 + \cdots , \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots , \tag{4}$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots . \tag{5}$$

- Series (3) diverges, i.e., it fails to converge for all $x \neq 0$.
- Series (4) converges for all x.
- Series (5) converges for $|x| < 1$ and diverges for $|x| > 1$.

Radius of convergence

- Some power series behave like (3) and converge only for $x = 0$, we are not interested in these.
- Some others, like (4), converge for all x and they are obviously the easiest ones.
- All other series are more or less like (5).

Thus, we observe that: To each series of the kind (5), there corresponds a positive real number R, called the **radius of convergence**, with the property that the series converges if $|x| < R$ and diverges if $|x| > R$. [$R = 1$ for the series (5)]

It is customary to put R equal to 0 when the series converges only for $x = 0$ and equal to ∞ when it converges for all x. We can cover all the possibilities by stating:
Definition: Each power series in x has a radius of convergence R, where $0 \leq R \leq \infty$, with the property that the series converges if $|x| < R$ and diverges if $|x| > R$.

□

Note that if $R = 0$, no x satisfies $|x| \leq R$, and if $R = \infty$, then no x satisfies $|x| > R$.

Radius of convergence

Let $\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$ be a series of nonzero constants. Recall that if the limit

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = L$$

exists, then the ratio tests assert that the series converges if $L < 1$ and diverges if $L > 1$.

For our power series (1), for each $a_n \neq 0$, $x \neq 0$, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = L.$$

Then (1) converges if $L < 1$ and diverges if $L > 1$.

Radius of convergence

We write

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|, \quad \text{if this limit exists.}$$

$$\left(R = \infty \text{ if the limit of } \left| \frac{a_n}{a_{n+1}} \right| \to \infty \right).$$

If R is infinite and nonzero, then it determines an interval of convergence $-R < x < R$ such that inside the interval, the series converges and outside the interval, it diverges. A power series may or may not converge at either endpoint of its interval of convergence.
(Similar thing may happen for a Fourier series.)

Suppose that series (1) converges for $|x| < R$ with $R > 0$ and denote its sum by $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (6)$$

Then $f(x)$ is automatically continuous and has derivatives for $|x| < R$. The series can be differentiated term-wise:

$$\begin{aligned} f^{'}(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\ f^{''}(x) &= 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \end{aligned}$$

and so on.

Each of the resulting series converges for $|x| < R$.

Continuing this way, we can link a_n's to $f(x)$ and its derivatives by

$$f^{(n)}(0) = n!\, a_n \quad \Rightarrow \quad a_n = \frac{f^{(n)}(0)}{n!}. \qquad (7)$$

$f(x)$ can be expressed as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f^{'}(0)x + \frac{f^{''}(0)}{2!} x^2 + \cdots .$$

Also note that the series (6) can be integrated term-wise provided the limits of integration lie inside the interval of convergence.

Definition:

A function with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{8}$$

is valid in some neighbourhood of the point x_0 is said to be analytic at x_0. In this case a_n's are necessarily given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

The series (8) is called the Taylor series of $f(x)$ about $x = x_0$.

Functions $e^x, \sin x$ and $\cos x$ are always analytic at $x_0 = 0$. The series seen on the LHS in

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

are the Taylor series of these functions at this point.

Most of the required information on analyticity can be obtained from the following facts:

- All polynomials and the functions $e^x, \sin x$ and $\cos x$ are analytic at all points.
- If $f(x)$ and $g(x)$ are analytic at x_0, then $f(x) + g(x), f(x)g(x)$, and $f(x)/g(x)$ (for $g(x_0) \neq 0$) are also analytic at x_0.
- If $f(x)$ is analytic at x_0 and $f^{-1}(x)$ is a continuous inverse, then $f^{-1}(x)$ is analytic at $f(x_0)$ if $f'(x_0) \neq 0$.
- If $g(x)$ is analytic at x_0 and $f(x)$ is analytic at $g(x_0)$, then $f(g(x))$ is analytic at x_0.
- The sum of a power series is analytic at all points inside the interval of convergence.

Power series solution for first order ODEs

Let us first start with a very simple first order ODE and try to compare the result obtained by elementary method.

Consider the ODE

$$y' - y = 0. \tag{9}$$

Assume that (9) has a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{10}$$

that converges for $|x| < R$ with $R > 0$.

In other words,

we assume that (9) has a solution that is analytic at the origin.

Power series solution for first order ODEs

By term-by-term differentiation,

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + (n+1)a_{n+1}x^n + \cdots . \tag{11}$$

Since $y' = y$, the coefficients of powers of x must be equal, i.e.,

$$a_1 = a_0, a_1 = 2a_2, a_2 = 3a_3, \ldots, (n+1)a_{n+1} = a_n, \cdots$$

i.e.,

$$a_1 = a_0,\ a_2 = \frac{a_1}{2} = \frac{a_0}{2},\ a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \cdots,\ a_n = \frac{a_0}{n!}$$

Hence the power series solution for (9) is

$$y(x) = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right). \tag{12}$$

We can easily observe that the series in (12) is nothing but the power series expansion of e^x.

Power series solution for first order ODEs

Hence the solution can be recognized as a familiar elementary function and written as

$$y = a_0 e^x. \tag{13}$$

Now consider another first order equation:

$$xy' - (x+2)y + 2x^2 + 2x = 0 \tag{14}$$

By adopting a similar approach,

we can arrive at $a_0 = 0$ and $a_1 = 2$.

The recurrence formula for the coefficients would be

$$a_n = \frac{a_{n-1}}{n-2}.$$

Power series solution for first order ODEs

It gives

$$a_3 = a_2,\ a_4 = \frac{a_3}{2} = \frac{a_2}{2},\ a_5 = \frac{a_4}{3} = \frac{a_2}{6}, \cdots .$$

The solution can be written as

$$y(x) = 2x + a_2\left(x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{6} + \frac{x^6}{24} + \cdots\right) = 2x + a_2x^2e^x \tag{15}$$

Procedure:

1. We assume a series solution: $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
2. Differentiate y, and put y and y' in the equation.
3. Equate the coefficients of various powers of x to get a relation between a pair of coefficients.
4. Get all non-zero coefficients in terms of one coefficient.
5. Put them back in $y(x) = \sum_{n=0}^{\infty} a_n x^n$ to get the solution.