MA 542 Differential Equations

Lecture 7

(January 18, 2022)

II. Method of variation of parameters

Method of variation of parameters, a method based on the availability of the complementary function of a differential equation, is considered to be the most powerful method for finding a particular solution. This method is due to the great mathematician Lagrange.

Suppose y_1 and y_2 are the two particular linearly independent solutions for the homogeneous equation $y'' + py' + qy = 0$. Then its complementary function can be written as

$$y = Ay_1 + By_2 \tag{1}$$

where A and B are arbitrary constants.

Assume that A and B in (1) are not constants, but instead are functions of x, denoted by $A(x)$ and $B(x)$, respectively. Then consider that

$$y = A(x)y_1 + B(x)y_2 \tag{2}$$

is a complete solution of the non-homogeneous equation

$$y'' + py' + qy = R(x). \tag{3}$$

Since the method assumes that the quantities A and B vary with respect to x, the method is generally known as the method of variation of parameters.

It is obvious that, since we have two unknowns to be determined, we usually require two conditions: One of these conditions arises from the fact that the assumed solution must satisfy the differential equation and the second one is at our disposal.

Thus differentiating (2) once, we get

$$y' = A(x)y_1' + B(x)y_2' + (A'(x)y_1 + B'(x)y_2). \tag{4}$$

We realize that further differentiation would introduce second derivatives of the unknown variables.

We choose the following condition which simplifies (4):

$$A'(x)y_1 + B'(x)y_2 = 0. \tag{5}$$

Then (4) reduces to

$$y' = A(x)y_1' + B(x)y_2'. \tag{6}$$

Differentiating (6),

$$y'' = A(x)y_1'' + B(x)y_2'' + A'(x)y_1' + B'(x)y_2'. \tag{7}$$

Substituting the values of y, y' and y'' into (3), we obtain

$$(Ay_1'' + By_2'' + A'y_1' + B'y_2') + p(Ay_1' + By_2') + q(Ay_1 + By_2) = R(x). \tag{8}$$

But, since y_1 and y_2 are solutions of the homogeneous equation, therefore

$$y_1'' + py_1' + qy_1 = 0,$$
$$y_2'' + py_2' + qy_2 = 0.$$

Hence (8) reduces to

$$A'y_1' + B'y_2' = R(x). \tag{9}$$

Thus, we have obtained two conditions (5) and (9) in two unknowns A' and B':

$$A'y_1 + B'y_2 = 0,$$
$$A'y_1' + B'y_2' = R(x).$$

Using Cramer's rule, we obtain the solutions for A' and B' as

$$A' = \frac{\begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 R}{y_1 y_2' - y_1' y_2} = \frac{-y_2 R}{W(y_1, y_2)},$$

$$B' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & R(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 R}{W(y_1, y_2)}.$$

Integrating the above two equations, we get

$$A(x) = -\int \frac{y_2 R}{W}\, dx + c_1, \tag{10a}$$

$$B(x) = \int \frac{y_1 R}{W}\, dx + c_2. \tag{10b}$$

Hence the complete solution can be written as

$$\begin{aligned} y &= A(x)y_1 + B(x)y_2 \\ &= c_1y_1 + c_2y_2 + \left(-y_1\int \frac{y_2R}{W}\,dx\right) + \left(y_2\int \frac{y_1R}{W}\,dx\right). \end{aligned} \tag{11}$$

In (11) above, the first two terms on the right side represent the complementary function and the third and fourth terms represent the particular integral.

Example:

$$y'' + y = \sec x.$$

Solution:

A complementary function of the given equation is $y = A\cos x + B\sin x$.

In order to use the method of variation of parameters, we assume

$$y = A(x)\cos x + B(x)\sin x.$$

The following two equations are to be solved for A' and B':

$$A' \cos x + B' \sin x = 0,$$
$$-A' \sin x + B' \cos x = \sec x.$$

Solving for A' and B' by Cramer's rule:

$$A' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = -\sin x \sec x = -\frac{\sin x}{\cos x},$$

$$B' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \cos x \sec x = 1.$$

Integrating with respect to x,

$$A = -\int \frac{\sin x}{\cos x}\, dx + c_1 = \ln(\cos x) + c_1,$$

$$B = \int dx + c_2 = x + c_2,$$

where c_1 and c_2 are two arbitrary constants.

The complete solution can be written as

$$\begin{aligned} y &= (c_1 + \ln(\cos x))\cos x + (c_2 + x)\sin x \\ &= [c_1 \cos x + c_2 \sin x] + [\cos x \ln(\cos x) + x \sin x]. \end{aligned}$$

In the above solution, the first part corresponds to the complementary function whereas the second part corresponds to the particular integral.

Exercise

$$y'' + y' - 2y = \ln x.$$

Exercise

$$x^2 y'' + xy' - y = x^2 e^x, \quad \text{given that } y = x \text{ is a solution.}$$

III. Differential operator method

Consider the n-th order nonhomogeneous linear differential equation with constant coefficients as:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x) \tag{12}$$

where $a_n, a_{n-1}, \ldots, a_0$ are all constants.

Now defining the differential operator as

$$D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, \ldots, D^n = \frac{d^n}{dx^n},$$

we can write (12) in a compact form as follows:

$$\phi(D)y = f(x) \tag{13}$$

where $\phi(D)$ is a linear polynomial operator in D and is given by

$$\phi(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0. \tag{14}$$

Now treating the equation (13) as an algebraic equation, we can solve for y:

$$y = \frac{1}{\phi(D)}\, f(x) \tag{15}$$

We define the inverse operator $1/D$ as

$$\frac{1}{D} = \int \tag{16}$$

Similarly,

$$\frac{1}{D^2} = \int\int, \dots, \frac{1}{D^n} = \int\int\cdots\int \ (n \text{ integrals}).$$

The following results will give ideas to solve differential equations by operator method.

Result 1:

$$\phi(D)e^{ax} = e^{ax}\phi(a)$$

Proof: Since $De^{ax} = ae^{ax}, D^2e^{ax} = a^2e^{ax}, \dots, D^ne^{ax} = a^ne^{ax}$, then

$$\begin{aligned} \phi(D)e^{ax} &= (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)e^{ax} \\ &= (a_na^n + a_{n-1}a^{n-1} + \cdots + a_1a + a_0)e^{ax} = e^{ax}\phi(a). \end{aligned}$$

Corollary

$$\frac{1}{\phi(D)}\, e^{ax} = \frac{e^{ax}}{\phi(a)},\ \ \phi(a) \neq 0$$

Result 2

For the case $\phi(a) = 0$,

$$\phi(D)(e^{ax}V) = e^{ax}\phi(D+a)V$$

where V is a function of x.

Result 3

$$\phi(D^2)\left\{\begin{array}{c}\cos ax\\ \sin ax\end{array}\right\} = \phi(-a^2)\left\{\begin{array}{c}\cos ax\\ \sin ax\end{array}\right\}$$

where

$$\phi(D^2) = a_n D^{2n} + a_{n-1}D^{2n-2} + \cdots + a_1 D^2 + a_0.$$

Note:

The method of operators is very powerful for differential equations with constant coefficients and with $f(x)$ of the type $k, x^n, e^{ax}, \cos ax, \sin ax$ or any combination with these functions.

Exercise

$$4y'' + 12y' + 9y = 144e^{\frac{-3x}{2}}.$$

Solution:

$$y = (A + Bx)e^{\frac{-3x}{2}} + 6x^3.$$

Exercise

$$y'' - 5y' + 6y = 100 \sin 4x$$

Solution:

$$y = Ae^{2x} + Be^{3x} + 4 \cos 4x - 2 \sin 4x.$$

Exercise

$$y'' + 9y = \cos 3x.$$

Linear equations with variable coefficients

Now we are going to discuss a method for solving a certain class of linear differential equations with variable coefficients of the following form:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = f(x). \tag{17}$$

This equation is called Euler-Cauchy differential equation and can be solved by transforming it into a linear differential equation with constant coefficients.
Here $a_n, a_{n-1}, \ldots, a_1, a_0$ are all constants and the coefficient of derivative of each order is a constant multiple of corresponding power of the independent variable.

However, this method is not applicable for a general linear differential equation with arbitrary variable coefficients of the following form:

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = f(x). \tag{18}$$

The change of independent variable for equation (17) is defined by $x = e^z$ or $z = \ln x, \ x \neq 0$.
Denote $D = \dfrac{d}{dx}, D^2 = \dfrac{d^2}{dx^2}$ and so on.

Subsequently,

$$
\begin{aligned}
D &= \frac{d}{dx} = \frac{dz}{dx}\frac{d}{dz} = \frac{1}{x}\frac{d}{dz} \\
D^2 &= \frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{d}{dz}\right) = -\frac{1}{x^2}\frac{d}{dz} + \frac{1}{x}\frac{dz}{dx}\frac{d^2}{dz^2} = -\frac{1}{x^2}\frac{d}{dz} + \frac{1}{x^2}\frac{d^2}{dz^2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
xD &= \frac{d}{dz} = D_1 \\
x^2D^2 &= -\frac{d}{dz} + \frac{d^2}{dz^2} \\
&= -D_1 + D_1^2 = D_1(D_1 - 1).
\end{aligned}
$$

Hence by the substitution, we get a differential equation with constant coefficients in terms of the transformed variable. After solving and using $z = \ln x$, we get the required solution for y.

Example:

$$x^2y'' + xy' = 12x \ln x.$$

Solution:

By making the substitution $x = e^z$, we have $z = \ln x$ and $\dfrac{dz}{dx} = \dfrac{1}{x}$.

The equation gets converted to

$$[D_1(D_1 - 1) + D_1]y = 12ze^z.$$

i.e.,

$$\frac{d^2y}{dz^2} = 12ze^z.$$

We get a complementary function as

$$y_c = A + Bz.$$

We get the particular integral from

$$\begin{aligned} y_p &= \frac{12}{D_1^2} ze^z \\ &= 12e^z \frac{1}{(D_1+1)^2} z \\ &= 12e^z (1+D_1)^{-2} z \\ &= 12e^z (1 - 2D_1 + 3D_1^2 - \cdots) z \\ &= 12e^z (z-2). \end{aligned}$$

Therefore, a complete solution is

$$\begin{aligned} y &= A + Bz + 12e^z(z-2) \\ &= A + B \ln x + 12x(\ln x - 2). \end{aligned}$$