MA 542 Differential Equations
Lecture 6
(January 17, 2022)

The homogeneous equation with constant coefficients:

Consider the coefficients $P(x)$ and $Q(x)$ to be constants in equation $y'' + P(x)y' + Q(x)y = 0$:

$$y'' + py' + qy = 0. \tag{1}$$

The fact that the exponential function e^{mx} has the property that its derivatives are all constant multiples of the function itself will lead us to consider

$$y = e^{mx} \tag{2}$$

as a possible solution for (1) if the constant m is suitably chosen. This also ensures that we always get a non-zero solution for all values of x.

Since $y' = me^{mx}$ and $y'' = m^2 e^{mx}$,

substituting these in (1) yields

$$(m^2 + pm + q)e^{mx} = 0. \tag{3}$$

Since e^{mx} is never zero, (3) holds if and only if m satisfies the *auxiliary equation*

$$m^2 + pm + q = 0. \tag{4}$$

There will be two roots of (4) in the form $m_{1,2} = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$ and three cases will arise:

Case I: Distinct real roots

The roots m_1 and m_2 are distinct and real if and only if $p^2 - 4q > 0$. In this case, we get the solutions as $e^{m_1 x}$ and $e^{m_2 x}$ giving us the general solution as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{5}$$

Case II: Equal real roots

The roots m_1 and m_2 will be real and equal if and only if $p^2 = 4q$. Here we obtain only one solution $y_1 = e^{mx}$ with $m = -p/2$.

However, a second solution can easily be found to be $y_2 = xy_1$. Hence, in this case the general solution for (1) will be

$$y = c_1 e^{mx} + c_2 x\, e^{mx} = (c_1 + c_2 x) e^{mx}. \tag{6}$$

Case III: Pair of complex conjugate roots

The two roots m_1 and m_2 are complex if and only if $p^2 - 4q < 0$. Then m_1 and m_2 can be written in the form $a \pm \mathrm{i}b$.

Then solutions for (1) can be written as

$$e^{m_1 x} = e^{(a+\mathrm{i}b)x} = e^{ax}(\cos bx + \mathrm{i}\sin bx) \tag{7}$$

and

$$e^{m_2 x} = e^{(a-\mathrm{i}b)x} = e^{ax}(\cos bx - \mathrm{i}\sin bx) \tag{8}$$

Since we are interested only in solutions that are real-valued functions, we can add (7) and (8) and divide by 2; subtract (8) from (7) and divide by 2i, to obtain, respectively,

$$e^{ax}\cos bx \quad \text{and} \quad e^{ax}\sin bx \tag{9}$$

So, the general solution in this case can be written as

$$y = e^{ax}(c_1\cos bx + c_2\sin bx). \tag{10}$$

Example:

$$y'' + 7y' - 18y = 0.$$

Solution:

The roots of the auxiliary equation are 2 and -9. Hence the general solution is

$$y = c_1 e^{2x} + c_2 e^{-9x}.$$

Example:

$$y'' + 8y' + 16y = 0.$$

Solution:

The (repeated) root of the auxiliary equation is -4. Hence the general solution is

$$y = (c_1 + c_2 x)e^{-4x}.$$

Example:

$$y'' + 2y' + 5y = 0.$$

Solution:

The pair of complex roots of the auxiliary equation is $-1 \pm 2\mathrm{i}$. Hence the general solution is

$$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x).$$

It is clear that the qualitative nature of this general solution of the differential equation is fully determined by the signs and relative magnitudes of the coefficients p and q, and can be radically changed by altering their numerical values. All the above ideas are due to the great mathematician Euler.

Higher order homogeneous equations:

A linear homogeneous differential equation with constant coefficients and of order $n > 2$ has the form:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, n > 2, a_0 \neq 0. \tag{11}$$

Assuming a trial solution $y = Ae^{mx}$ leads to the auxiliary equation

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \tag{12}$$

The degree of this algebraic equation will be the same as the order of the differential equation (11).

We will get the roots of (12) as $m_1, m_2, \ldots, m_n$ and get the solutions depending on the nature of the roots since some of the roots may be real and unequal, some may be real and equal and some may be complex.

Example:

$$y''' - 6y'' + 12y' - 8y = 0.$$

Solution:

The auxiliary equation $m^3 - 6m^2 + 12m - 8 = 0$ gives the root as 2 of multiplicity 3. Hence the general solution can be written as

$$y = (c_1 + c_2 x + c_3 x^2)e^{2x}.$$

Example:

$$y''' + 2y'' - y' - 2y = 0.$$

Solution:

The auxiliary equation $m^3 + 2m^2 - m - 2 = 0$ gives the roots as $-2, 1$ and -1. Hence the general solution can be written as

$$y = c_1 e^{-2x} + c_2 e^{x} + c_3 e^{-x}.$$

The solution of nonhomogeneous second-order equation

In the preceding lectures, we have discussed how to find solutions of the second-order homogeneous differential equation. Now we try to find solution of the general linear second-order nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = R(x). \tag{13}$$

Theorem

If y_p is any specific solution of (13) and $y_c = c_1y_1 + c_2y_2$ is a complete solution of the corresponding homogeneous equation (i.e., $R(x) \equiv 0$), then a complete solution of the nonhomogeneous equation (13) is

$$y = y_c + y_p.$$

In order to obtain the solution y_p, called a particular integral, there are three methods as follows:

1. Method of undetermined coefficients,
2. Method of variation of parameters,
3. Method of operators.

I. Method of undetermined coefficients

This method is a procedure for finding y_p when (13) has the form

$$y'' + py' + qy = R(x), \tag{14}$$

where p and q are constants and $R(x)$ is an exponential, a sine or cosine, a polynomial, or some combination of such functions.

Let us study the equation

$$y'' + py' + qy = e^{ax}. \tag{15}$$

It is natural to guess that

$$y_p = Ae^{ax} \tag{16}$$

might be a particular solution of (15).

Here A is the undetermined coefficient that we want to determine in such a way that (16) will actually satisfy (15).

On substituting (16) into (15), we get

$$A(a^2 + pa + q)e^{ax} = e^{ax},$$

so that

$$A = \frac{1}{a^2 + pa + q}. \tag{17}$$

This value of A will make (16) a particular solution of (15) except when the denominator on the right side of (17) is zero. That is, the exception arises when a is a root of the auxiliary equation

$$m^2 + pm + q = 0. \tag{18}$$

For this, we may have to try another solution like $y = Axe^{ax}$ and so on.

The main points to follow while trying to solve an equation of the type (15) are

1. If a is not a root of the auxiliary equation (18), then (15) has a particular solution of the form Ae^{ax}.
2. If a is a simple root of the auxiliary equation (18), then (15) has a particular solution not of the form Ae^{ax} but does have one of the form Axe^{ax}.
3. If a is a double root of the auxiliary equation (18), then (15) has a particular solution not of the form Axe^{ax} but does have one of the form Ax^2e^{ax}.

Another important case where the method of undetermined coefficients can be applied is that in which the right side of (15) is replaced by $\sin bx$:

$$y'' + py' + qy = \sin bx. \tag{19}$$

A trial solution can be taken as

$$y_p = A \sin bx + B \cos bx. \tag{20}$$

Even if the right side is replaced by $\cos bx$ or any linear combination of $\sin bx$ and $\cos bx$, the method will still work.

But if y_p given by (20) satisfies the homogeneous equation corresponding to (19), then this method will no longer work and we have to look for a new trial solution of the form:

$$y_p = x(A \sin bx + B \cos bx) \tag{21}$$

instead of the form in (20).

Example

Find a particular solution of $y'' + 5y' + 6y = 12$.

Solution:

The right side of the equation is 12, i.e., $12e^{0x}$ with $a = 0$. Therefore, the trial solution is $y_p = 12A$ where $A = 1/(a^2 + ap + q) = 1/6$.
Therefore $y_p = 2$.

Hence the general solution can be written as

$$y = c_1 e^{-2x} + c_2 e^{-3x} + 2.$$

Example:

Find a particular solution of $y'' + y = \sin x$

Solution:

$y_p = A\sin x + B\cos x$ cannot be taken as a trial solution since it satisfies the homogeneous equation. Hence the trial solution has to be $y_p = x(A\sin x + B\cos x)$. Putting this in the equation, we will get

$$2A\cos x - 2B\sin x = \sin x.$$

On comparing the coefficients of $\cos x$ and $\sin x$ on both sides, we get $A = 0$ and $B = -1/2$. Hence the particular integral is given by

$$y_p = -\frac{x}{2}\cos x.$$

Hence the general solution can be written as

$$y = c_1\cos x + c_2\sin x - \frac{x}{2}\cos x.$$