MA 542 Differential Equations
Lecture 5
(January 13, 2022)

Boundary conditions are conditions prescribed on the boundary.

Boundary may be a boundary with respect to any of the independent variables.

Initial conditions are conditions prescribed at one point only.

These conditions are in terms of the value of some form of the dependent variable (may be in terms of its derivatives too) at some specific value of the independent variable.

The main component of this type of problems is what is called Governing Equation.

With respect to ODEs,

we can have only boundary conditions or only initial conditions, not both for the same problem.

They are, respectively, called boundary value problems or initial value problems.

However, with respect to PDEs (evolution equations)

we may have both boundary conditions and initial conditions for the same problem.

This type of problems are called

Initial Boundary Value Problems (IBVP).

Time-independent PDEs

form Boundary Value Problems, not IBVPs.

A boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions.
Depending on the order of the equation, those conditions are prescribed on the boundary in terms of the values of the variables or its derivative(s).

In other words, a solution to a BVP is a solution to the differential equation which also satisfies the boundary conditions.

To be useful in applications, a BVP should be well-posed. This means that given the input to the problem there exists a unique solution, which depends continuously on the input.

An initial value problem (IVP) consists of a differential equation and a set of conditions to be satisfied at the initial value of the independent variable or its derivative(s) (for ODE) or at that of one of the independent variables or its derivative(s) (for PDE).

A more mathematical way to picture the difference between a BVP and an IVP is

an IVP has all of the conditions specified at the same value of the independent variable in the equation (and that value is at the lower value of the boundary of the domain, thus the term 'initial' value), while a BVP has conditions specified at the extremes of the independent variable(s).

For example

for a second-order differential equation

if the independent variable is time over the domain $[0, 1]$, an IVP would specify a value of $y(t)$ and $y'(t)$ at time $t = 0$, to be precise, the initial conditions will be something like $y(0) = \alpha, y'(0) = \beta$.

On the other hand

a BVP would specify values for $y(t)$ (or its derivatives) at both $t = 0$ and $t = 1$, to be precise, the boundary conditions will be something like $y(0) = \alpha_1, y(1) = \beta_1$ or $y'(0) = \alpha_2, y'(1) = \beta_2$.

Theorem

Let $y_1(x)$ and $y_2(x)$ be any two solutions of equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{1}$$

on $[a, b]$. Then their Wronskian $W = W(y_1, y_2) = y_1 y_2' - y_1' y_2$ is either identically zero or never zero on $[a, b]$.

Proof:

First we begin with the following observation:

$$\begin{aligned} W' &= y_1 y_2'' + y_1' y_2' - y_2 y_1'' - y_2' y_1' \\ &= y_1 y_2'' - y_2 y_1''. \end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are both solutions of (1), we have

$$y_1'' + P y_1' + Q y_1 = 0, \tag{2}$$

$$y_2'' + P y_2' + Q y_2 = 0. \tag{3}$$

On multiplying (2) by y_2 and (3) by y_1 and subtracting

$$(y_1 y_2^{''} - y_2 y_1^{''}) + P(y_1 y_2^{'} - y_2 y_1^{'}) = 0$$

which can be written as

$$\frac{dW}{dx} + PW = 0.$$

The solution of this equation can be obtained as

$$W = ce^{-\int P dx}.$$

Since the exponential factor is never zero, we see that W is identically zero if the constant $c = 0$, and never zero if $c \neq 0$.
This completes the proof.

Theorem:

Let $y_1(x)$ and $y_2(x)$ be two solutions of equation (1) on the interval $[a, b]$. Then they are linearly dependent on this interval if and only if their Wronskian $W = W(y_1, y_2) = y_1 y_2^{'} - y_2 y_1^{'}$ is identically zero.

Proof:

We begin by assuming that $y_1(x)$ and $y_2(x)$ are linearly dependent and show that it leads to $y_1 y_2^{'} - y_2 y_1^{'} = 0$.

If both or one of the functions are(is) identically zero, then the conclusion is obvious. We, therefore, assume that neither of them is identically equal to zero and are such that linear dependence allows us to write each to be a constant multiple of the other.

By writing $y_2 = c y_1$ for some constant c, we have $y_2^{'} = c y_1^{'}$. This leads to

$$y_1 y_2^{'} - y_2 y_1^{'} = y_1 (c y_1^{'}) - (c y_1) y_1^{'} = 0,$$

which proves the first half of the theorem.

On the contrary, assume that their Wronskian W is identically equal to zero and establish their linear dependence. If y_1 is identically zero on $[a, b]$, then the functions are linearly dependent.

Therefore, we may assume that y_1 does not vanish identically on $[a, b]$ from which it follows by continuity that y_1 does not vanish at all on some sub-interval $[c, d]$ of $[a, b]$.

Since the Wronskian is identically equal to zero on $[a, b]$, we can divide it by y_1^2 to get

$$\frac{y_1 y_2^{'} - y_2 y_1^{'}}{y_1^2} = 0$$

on $[c, d]$.

The above can be written in the form $(y_2/y_1)^{'} = 0$ which on integration gives us $y_2/y_1 = k$ or $y_2(x) = ky_1(x)$ for some constant k and all x in $[c, d]$.

Finally, since $y_2(x)$ and $ky_1(x)$ have equal values in $[c, d]$, they have equal derivatives there as well and hence we can conclude that

$$y_2(x) = ky_1(x)$$

for all x in $[a, b]$ implying that $y_1(x)$ and $y_2(x)$ are linearly dependent in $[a, b]$ which completes the proof.

Example:

Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval, and find the particular solution for which $y(0) = 2$ and $y'(0) = 3$.

Solution:

It can be easily verified that $y_1(x) = \sin x$ and $y_2 = \cos x$ are solutions of the given equation. Their linear independence on any interval $[a, b]$ follows from the either of the following:

- $y_1/y_2 = \tan x$ is not a constant,
- Their Wronskian never vanishes:

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1.$$

Since $P(x) = 0$ and $Q(x) = 1$ are continuous on $[a, b]$, it follows that $y = c_1 \sin x + c_2 \cos x$ is the general solution of the given equation on $[a, b]$.

Further, since the interval $[a, b]$ can be expanded indefinitely without introducing points at which $P(x)$ or $Q(x)$ is discontinuous, this general solution is valid for all x.

For finding the required particular solution, we use the conditions $y(0) = 2$ and $y'(0) = 3$ to get and solve the following system:

$$c_1 \sin 0 + c_2 \cos 0 = 2,$$
$$c_1 \cos 0 - c_2 \sin 0 = 3,$$

which yields $c_2 = 2$ and $c_1 = 3$.

Therefore

$$y = 3 \sin x + 2 \cos x$$

is the particular solution that satisfies the given conditions.

Note:

The concepts of linear dependence and independence has much wider implication than what is observed here. The best examples are found in linear algebra, algebra, geometry and analysis.

The use of a known solution to find another:

Recall the general second-order linear equation:

$$y'' + P(x)y' + Q(x)y = 0. \tag{4}$$

We know that

it is easy to write down the solution of equation (4) whenever we know two linearly independent solutions $y_1(x)$ and $y_2(x)$.

If $y_1(x)$ is given to be one of the solutions, then the second solution can be found from the following relation:

$$y_2 = y_1 \int \frac{1}{y_1^2}\, e^{-\int P\, dx}\, dx \tag{5}$$

This relation can be obtained in the following manner:

Let $y_2 = vy_1$ to be a solution of (4) so that

$$y_2^{''} + Py_2^{'} + Qy_2 = 0. \tag{6}$$

Then equation (6) becomes

$$v(y_1^{''} + Py_1^{'} + Qy_1) + v^{''}y_1 + v'(2y_1^{'} + Py_1) = 0.$$

Since y_1 is a solution of (4), we have $y_1^{''} + Py_1^{'} + Qy_1 = 0$ and hence this reduces to

$$v^{''}y_1 + v'(2y_1^{'} + Py_1) = 0$$

or,

$$\frac{v^{''}}{v'} = -2\,\frac{y_1^{'}}{y_1} - P.$$

On integration,

$$\ln v' = -2\ln y_1 - \int P\,dx$$

so that

$$v' = \frac{1}{y_1^2}\,e^{-\int P\,dx}.$$

Therefore,

$$v = \int \frac{1}{y_1^2}\, e^{-\int P\, dx}\, dx$$

This gives the second solution y_2 as

$$y_2 = y_1 \int \frac{1}{y_1^2}\, e^{-\int P\, dx}\, dx.$$

By showing that y_1 and $y_2 = vy_1$ are linearly independent, the general solution can be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Example:

If $y_1 = x$ is a solution of $x^2 y'' + xy' - y = 0$, then find the general solution.

Solution: The given equation can be written as

$$y'' + \frac{1}{x}\, y' - \frac{1}{x^2}\, y = 0$$

with $P(x) = 1/x$.

This gives the second solution y_2 as

A second linearly independent solution is given by $y_2 = vy_1$, where

$$v = \int \frac{1}{x^2}\, e^{-\int (1/x)\, dx}\, dx = \int \frac{1}{x^2}\, e^{-\ln x}\, dx = \int x^{-3}\, dx = \frac{x^{-2}}{-2}.$$

This yields

$y_2 = vy_1 = (-\dfrac{1}{2x^2})x = (-1/2)\dfrac{1}{x}$ so that the general solution is given by

$$y = c_1 x + c_2 \frac{1}{x}.$$