MA542 Differential Equations

Lecture 45

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Green's Function for PDEs

Green's Function for Poisson's equation

We are going to analyze Green's function for Poisson's equation, a time-independent partial differential equation,

$$\mathcal{L}u = f, \tag{1}$$

where $\mathcal{L} = \nabla^2$, the Laplacian. Initially we will assume that u satisfies homogeneous boundary conditions and that the region is finite.

The following Green's formula will be frequently used, either in its three- or two-dimensional forms:

$$\begin{aligned} \iiint (u\nabla^2 v - v\nabla^2 u)dV &= \oiint (u\nabla v - v\nabla u) \cdot \hat{n}\, dS, & (2) \\ \iint (u\nabla^2 v - v\nabla^2 u)dA &= \oint (u\nabla v - v\nabla u) \cdot \hat{n}\, ds. & (3) \end{aligned}$$

Multidimensional Dirac delta function and Green's function

The Green's function is defined as the solution to the non-homogeneous problems with concentrated source, subject to homogeneous boundary conditions. We define a two-dimensional Dirac delta function as the product of two one-dimensional Dirac delta functions.

Green's Function for PDEs

If the source is concentrated at $\mathbf{x} = \mathbf{x_0}$ ($\mathbf{x} = x\hat{i} + y\hat{j}$, $\mathbf{x_0} = x_0\hat{i} + y_0\hat{j}$), then

$$\delta(\mathbf{x} - \mathbf{x_0}) = \delta(x - x_0)\delta(y - y_0). \tag{4}$$

Similar ideas hold in three dimensions. The fundamental property of this multidimensional Dirac delta function is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x_0})dA = f(\mathbf{x_0}). \tag{5}$$

When opened up, this becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\delta(x - x_0)\delta(y - y_0)dxdy = f(x_0, y_0). \tag{6}$$

In order to solve the non-homogeneous partial differential equation

$$\nabla^2 u = f(\mathbf{x}), \tag{7}$$

subject to homogeneous conditions along the boundary, we introduce the Green's function $G(\mathbf{x}, \mathbf{x_0})$ for Poisson's equation (it is also called the Green's function for Laplace's equation).

In other words,

$$\nabla^2 G(\mathbf{x}, \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}), \tag{8}$$

subject to the same homogeneous boundary conditions.

Here $G(\mathbf{x}, \mathbf{x_0})$ represents the response at $\mathbf{x}$ due to a source at $\mathbf{x_0}$.

Representation formula using Green's function

Green's formula, in its two-dimensional form, with $v = G(\mathbf{x}, \mathbf{x_0})$ becomes

$$\iint (u\nabla^2 G - G\nabla^2 u) dA = 0,$$

since both $u(\mathbf{x})$ and $G(\mathbf{x}, \mathbf{x_0})$ satisfy the same homogenous boundary conditions such that $\oint (u\nabla G - G\nabla u) \cdot \hat{n}\, ds$ vanishes.

From (7) and (8), it follows that

$$u(\mathbf{x_0}) = \iint f(\mathbf{x}) G(\mathbf{x}, \mathbf{x_0}) dA. \tag{9}$$

Green's Function for PDEs

If we reverse the roles of $\mathbf{x}$ and $\mathbf{x_0}$, we obtain

$$u(\mathbf{x}) = \iint f(\mathbf{x_0}) G(\mathbf{x_0}, \mathbf{x}) dA_0. \tag{10}$$

Symmetry

We here use Green's formula for the same problem with $G(\mathbf{x}, \mathbf{x_1})$ and $G(\mathbf{x}, \mathbf{x_2})$. Since both satisfy the same homogenous boundary conditions, we have

$$\iint [G(\mathbf{x}, \mathbf{x_1}) \nabla^2 G(\mathbf{x}, \mathbf{x_2}) - G(\mathbf{x}, \mathbf{x_2}) \nabla^2 G(\mathbf{x}, \mathbf{x_1})] \, dA = 0.$$

Since $\nabla^2 G(\mathbf{x}, \mathbf{x_1}) = \delta(\mathbf{x} - \mathbf{x_1})$ and $\nabla^2 G(\mathbf{x}, \mathbf{x_2}) = \delta(\mathbf{x} - \mathbf{x_2})$,

it follows, using the fundamental property of the Dirac delta function, that $G(\mathbf{x_1}, \mathbf{x_2}) = G(\mathbf{x_2}, \mathbf{x_1})$; the Green's function is symmetric, or rather we can say that it is unique.

Green's functions by the method of eigenfunction expansion

One method to obtain the Green's function for Poisson's equation in a finite region is to use an eigenfunction expansion.

We consider the related eigenfunctions:

$$\nabla^2 u = -\lambda u \tag{11}$$

subject to the same homogenous boundary conditions.

We assume that the eigenvalues and corresponding orthogonal eigenfunctions are known. Simple examples occur in rectangular and circular regions.

We attempt to solve for the Green's function $G(\mathbf{x}, \mathbf{x_0})$ from

$$\nabla^2 G(\mathbf{x}, \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}) \tag{12}$$

as an infinite series of eigenfunctions:

$$G(\mathbf{x}, \mathbf{x_0}) = \sum_{\lambda} a_\lambda u_\lambda(\mathbf{x}). \tag{13}$$

Here $u_\lambda(\mathbf{x})$ is the eigenfunction corresponding to the eigenvalue λ.

Since u_λ and $G(\mathbf{x}, \mathbf{x_0})$ satisfy the same homogeneous boundary conditions, we expect to be able to differentiate term-by-term:

$$\nabla^2 G = \sum_\lambda a_\lambda \nabla^2 u_\lambda = -\sum_\lambda a_\lambda \lambda u_\lambda(\mathbf{x}). \tag{14}$$

This can be verified by using Green's formula. Since $\nabla^2 G = \delta(\mathbf{x} - \mathbf{x_0})$, due to the multidimensional orthogonality of $u_\lambda(\mathbf{x})$, it follows that

$$-a_\lambda \lambda = \frac{\iint u_\lambda(\mathbf{x})\delta(\mathbf{x} - \mathbf{x_0})dA}{\iint u_\lambda^2(\mathbf{x})dA} = \frac{u_\lambda(\mathbf{x_0})}{\iint u_\lambda^2 dA}.$$

If $\lambda = 0$ is not an eigenvalue, then we can determine a_λ and subsequently the eigenfunction expansion of the Green's function:

$$G(\mathbf{x}, \mathbf{x_0}) = \sum_\lambda \frac{u_\lambda(\mathbf{x})u_\lambda(\mathbf{x_0})}{-\lambda \iint u_\lambda^2 dA}. \tag{15}$$

Green's Function for PDEs

This is the natural generalization of the one-dimensional result for Green's function corresponding to a non-homogenous Sturm-Liouville problem.

Example:

For a rectangle $0 < x < a$, $0 < y < b$ with boundary conditions zero on all four sides, it can be seen that the eigenvalues are $\lambda_{mn} = (n\pi/a)^2 + (m\pi/b)^2$, $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$ and the corresponding eigenfunctions are
$u_\lambda(\mathbf{x}) = \sin(n\pi x/a)\sin(m\pi y/b)$.

In this case the normalization constants are $u_\lambda^2 \, dx \, dy = (a/2)\cdot(b/2)$.

The Green's function can be expanded in a series of these eigenfunctions, a Fourier sine series in x and y,

$$G(\mathbf{x}, \mathbf{x_0}) = -\frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(n\pi x/a)\sin(m\pi y/b)\sin(n\pi x_0/a)\sin(m\pi y_0/b)}{(n\pi/a)^2 + (m\pi/b)^2}.$$

Direct Solution of Green's Functions

Green's functions can also be obtained by more direct methods. Consider the Green's function for Poisson's equation,

$$\nabla^2 G\delta(\mathbf{x}, \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}), \tag{16}$$

inside a rectangle $0 < x < a$, $0 < y < b$ with zero boundary conditions.

Instead of solving for this Green's function using a series of two-dimensional eigenfunctions (as done previously a while ago), we will use one-dimensional eigenfunctions, either a sine series in x or y due to the boundary conditions. Using a Fourier series in x,

$$G(\mathbf{x}, \mathbf{x_0}) = \sum_{n=1}^{\infty} A_n(y) \sin \frac{n\pi x}{a}. \tag{17}$$

By substituting (17) into (16), we obtain (since both $G(\mathbf{x}, \mathbf{x_0})$ and $\sin \dfrac{n\pi x}{a}$ satisfy the same set of homogenous boundary conditions),

$$\sum_{n=1}^{\infty} \left[\frac{d^2 A_n}{dy^2} - \left(\frac{n\pi}{a}\right)^2 A_n \right] \sin \frac{n\pi x}{a} = \delta(x - x_0)\delta(y - y_0).$$

Or

$$\begin{aligned}\frac{d^2 A_n}{dy^2} - \left(\frac{n\pi}{a}\right)^2 A_n &= \frac{2}{a}\int_0^a \delta(x-x_0)\delta(y-y_0)\sin\frac{n\pi x}{a}\,dx \\ &= \frac{2}{a}\sin\frac{n\pi x_0}{a}\,\delta(y-y_0). \qquad (18)\end{aligned}$$

The boundary conditions at $y = 0$ and $y = b$ imply that the Fourier coefficients must satisfy the corresponding boundary conditions

$$A_n(0) = 0 \quad \text{and} \quad A_n(b) = 0. \qquad (19)$$

Equation (18) with boundary conditions (19) may be solved by a Fourier sine series in y. On the other hand, since the non-homogeneous term for $A_n(y)$ is a one-dimensional Dirac delta function, we may solve (18) as we have done for Green's function.

The differential equation is homogeneous if $y \neq y_0$.

In addition if we utilize the boundary conditions, we obtain

$$A_n(y) = \begin{cases} c_n \sinh \frac{n\pi y}{a} \ \sinh \frac{n\pi(y_0 - b)}{a}, & y < y_0 \\ c_n \sinh \frac{n\pi(y-b)}{a} \ \sinh \frac{n\pi y_0}{a}, & y > y_0, \end{cases}$$

where in this form continuity at $y = y_0$ is automatically satisfied.

Again, we integrate (18) from y_{0-} to y_{0+} to obtain the jump in the derivative:

$$\left.\frac{dA_n}{dy}\right|_{y_{0-}}^{y_{0+}} = \frac{2}{a} \sin \frac{n\pi x_0}{a}$$

or

$$c_n \frac{n\pi}{a} \left[\sinh \frac{n\pi y_0}{a} \cosh \frac{n\pi(y_0 - b)}{a} - \sinh \frac{n\pi(y_0 - b)}{a} \cosh \frac{n\pi y_0}{a} \right] = \frac{2}{a} \sin \frac{n\pi x_0}{a} \tag{20}$$

Using an addition formula for hyperbolic functions we obtain

$$c_n = \frac{2 \sin(n\pi x_0 / a)}{n\pi \sinh(n\pi b / a)}.$$

This yields the Fourier sine series (in x) representation of the Green's function

$$G(\mathbf{x}, \mathbf{x_0}) = \sum_{n=1}^{\infty} \frac{2\sin(n\pi x_0/a)\sin(n\pi x/a)}{n\pi \sinh(n\pi b/a)} \begin{cases} \sinh \frac{n\pi(y_0-b)}{a} \sinh \frac{n\pi y}{a}, & y < y_0 \\ \sinh \frac{n\pi(y-b)}{a} \sinh \frac{n\pi y_0}{a}, & y > y_0. \end{cases} \tag{21}$$

The symmetry is exhibited explicitly.

Green's functions for problems with non-homogeneous boundary conditions:

The same methods used in the preceding sections can be used to solve Poisson's equation $\nabla^2 u = f(\mathbf{x})$ subject to non-homogeneous boundary conditions.

Consider

$$\nabla^2 u = f(\mathbf{x}) \tag{22}$$

with

$$u = h(\mathbf{x}) \tag{23}$$

on the boundary.

Green's Function for PDEs

The Green's function is defined by

$$\nabla^2 G(\mathbf{x}, \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}), \tag{24}$$

with

$$G(\mathbf{x}, \mathbf{x_0}) = 0 \tag{25}$$

for $\mathbf{x}$ on the boundary ($\mathbf{x_0}$ is often not on the boundary).

The Green's function satisfies the related homogeneous boundary conditions. To obtain the Green's function representation of the solution of (22) and (23), we again employ Green's formula:

$$\iint (u\nabla^2 G - G\nabla^2 u) dA = \oint (u\nabla G - G\nabla u) \cdot \hat{n}\ ds.$$

Using the defined differential equations and the boundary conditions,

$$\iint [u(\mathbf{x})\delta(\mathbf{x} - \mathbf{x_0}) - f(\mathbf{x})G(\mathbf{x}, \mathbf{x_0})] dA = \oint h(\mathbf{x})\nabla G \cdot \hat{n}\ ds,$$

and hence

$$u(\mathbf{x_0}) = \iint f(\mathbf{x})G(\mathbf{x}, \mathbf{x_0})] dA + \oint h(\mathbf{x})\nabla G \cdot \hat{n}\ ds.$$

We interchange $\mathbf{x}$ and $\mathbf{x_0}$, and we use the symmetry of $G(\mathbf{x}, \mathbf{x_0})$ to obtain

$$u(\mathbf{x}) = \iint f(\mathbf{x_0})G(\mathbf{x}, \mathbf{x_0})]dA_0 + \oint h(\mathbf{x_0})\nabla_{\mathbf{x_0}} G \cdot \hat{n}\, ds_0. \tag{26}$$

We must be very careful with the closed line integral, representing the effect of the non-homogenous boundary condition. $\nabla_{\mathbf{x_0}}$ is a symbol for the gradient with respect to the position of the source

$$\nabla_{\mathbf{x_0}} \equiv \frac{\partial}{\partial x_0}\hat{i} + \frac{\partial}{\partial y_0}\hat{j}.$$

Thus $G(\mathbf{x}, \mathbf{x_0})$ is the influence function for the source term while $\nabla_{\mathbf{x_0}} G \cdot \hat{n}$ is the influence function for the non-homogeneous boundary conditions.

Let us attempt to give an understanding to the influence function for the non-homogeneous boundary conditions $\nabla_{\mathbf{x_0}} G \cdot \hat{n}$. This is an ordinary derivative with respect to the source position in the normal direction.

Green's Function for PDEs

Using the definition of a directional derivative,

$$\nabla_{\mathbf{x_0}} G \cdot \hat{n} = \lim_{\nabla s \to 0} \frac{G(\mathbf{x}, \mathbf{x_0} + \Delta s \hat{n}) - G(\mathbf{x}, \mathbf{x_0})}{\Delta s}.$$

This yields an interpretation of this normal derivative of the Green's function. $(\mathbf{x}, \mathbf{x_0} + \Delta s \hat{n})/\Delta s$ is the response to a positive source of strength $1/\Delta s$ located at $\mathbf{x_0} + \Delta s \hat{n}$, while $-G(\mathbf{x}, \mathbf{x_0})/\Delta s$ is the response to a negative source (strength $-1/\Delta s$) located at $\mathbf{x_0}$.

The influence function for the non-homogeneous boundary condition consists of two concentrated sources of opposite effects whose strength is $1/\Delta s$ and distance apart is Δs, in the limit as $\Delta s \to 0$.

This is called a *dipole source*. Thus this non-homogeneous boundary condition has an equivalent effect as a surface distribution of dipoles.