MA542 Differential Equations

Lecture 44

27/04/2022

The general time-dependent heat conduction or diffusion problems are governed by equations of the type

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u - Q \tag{1}$$

subject to appropriate boundary conditions, may be homogeneous or non-homogeneous, and an initial condition. The solution is $u \equiv u(\bar{\mathbf{x}}, t)$ and α is the thermal diffusivity, $Q \equiv Q(\bar{\mathbf{x}}, t)$ a source of heat.

Irrespective of the type of the domain, which is bounded, the solution u for the problem has to satisfy all the related equations. The most natural and easily available time-independent PDEs are the elliptic ones, namely, Poisson's equation and Laplace's equation.

The steady-state problem, i.e., the time-independent one, is governed by

$$\nabla^2 u = Q, \tag{2}$$

which is Poisson's equation, and u and Q are functions of space coordinates only.

Green's Function for PDEs

The homogeneous form of Poisson's equation, i.e., when $Q = 0$, is Laplace's equation:

$$\nabla^2 u = 0. \tag{3}$$

While equation (**??**) is a standard parabolic equation, equations (**??**) and (**??**) are standard elliptic equations.

These type of equations arise in many physically interesting problems such as heat conduction and potential flow of fluids.

Let us first consider Poisson's equation (**??**) from the point of view of heat conduction.

Green's Function for PDEs

Without specifying the geometrical region, we assume that the temperature is specified on the boundary:

$$u = \beta,$$

where β is given and is usually not constant.

This problem is nonhomogeneous in two ways: due to the forcing term Q and the non-zero boundary condition β.

We can split the equilibrium temperature into two parts: $u = u_1 + u_2$, where u_1 is due to the forcing term and u_2 is due to the boundary condition:

$$\nabla^2 u_1 = Q, \quad u_1 = 0 \text{ on the boundary},$$
$$\nabla^2 u_2 = 0, \quad u_2 = \beta \quad \text{on the boundary}.$$

It can be easily verified that $u = u_1 + u_2$ satisfies Poisson's equation and the non-homogeneous BC. The problem for u_2 is the solution of Laplace's equation with non-homogeneous BC.

We focus our attention on Poisson's equation

$$\nabla^2 u_1 = Q$$

with homogeneous BC, i.e., $u_1 = 0$ on the boundary.

Since u_1 satisfies homogenous BC, we expect that the method of eigenfunction expansion is appropriate.

The problem can be analyzed in two different ways:
(1) we can expand the solutions in eigenfunctions of the related homogenous problem, coming from separation of variables of $\nabla^2 u_1 = 0$, or
(2) we can expand the solution in the eigenfunctions

$$\nabla^2 \phi + \lambda \phi = 0.$$

These two methods are different but related.

To be specific, let us consider the two-dimensional Poisson's equation in a rectangle $0 < x < a, 0 < y < b$ with zero conditions along all four boundaries, namely $x = 0, x = a, y = 0$ and $y = b$:

$$\nabla^2 u_1 = Q. \tag{4}$$

The related homogeneous problem, $\nabla^2 u_1 = 0$, which is Laplace's equation, can be separated in rectangular coordinates.

The method of eigenfunction expansion consists of expanding $u_1(x, y)$ in a series of these eigenfunctions:

$$u_1 = \sum_{n=1}^{\infty} B_n(y) \sin\left(\frac{n\pi x}{a}\right), \tag{5}$$

where the sine coefficients $B_n(y)$ are functions of y.

Our equation is

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = Q(x, y).$$

Differentiating (**??**) twice with respect to y and substituting this into Poisson's equation (**??**), we get

$$\sum_{n=1}^{\infty} \frac{d^2 B_n}{dy^2} \sin\left(\frac{n\pi x}{a}\right) + \frac{\partial^2 u_1}{\partial x^2} = Q. \tag{6}$$

$\dfrac{\partial^2 u_1}{\partial x^2}$ can be determined in two related ways: term-by-term differentiation with respect to x of the series (**??**) which is more direct or by Green's formula.

In either way, we obtain from (**??**),

$$\sum_{n=1}^{\infty} \left[\frac{d^2 B_n}{dy^2} - \left(\frac{n\pi}{a}\right)^2 B_n\right] \sin\left(\frac{n\pi x}{a}\right) = Q, \tag{7}$$

since both u_1 and $\sin(n\pi x/a)$ satisfy the same homogenous boundary conditions.

Thus the Fourier sine coefficients satisfy the following second-order ordinary differential equation:

$$\frac{d^2 B_n}{dy^2} - \left(\frac{n\pi}{a}\right)^2 B_n = \frac{2}{a}\int_0^a Q(x,y)\sin\left(\frac{n\pi x}{a}\right)\, dx \equiv q_n(y), \tag{8}$$

where the right-hand side is the sine coefficient of Q:

$$Q = \sum_{n=1}^{\infty} q_n \sin\left(\frac{n\pi x}{a}\right). \tag{9}$$

We must now solve (**??**).

We need two conditions for that. u satisfies Poisson's equation and the boundary conditions at $x=0$ and $x=a$. The boundary conditions at $y=0$ (for all x), $u_1=0$, and at $y=b$ (for all x), $u_1=0$, imply that

$$B_n(0)=0, \text{ and } B_n(b)=0. \tag{10}$$

Thus, the unknown coefficients in the method of eigenfunction expansion themselves solve a one-dimensional *non-homogeneous BVP*.

By using the methods of variation of parameters, the solution for $B_n(y)$ can be found as

$$
\begin{aligned}
B_n(y) &= \sinh\left(\frac{n\pi(b-y)}{a}\right)\int_0^y q_n(\xi)\sinh\left(\frac{n\pi\xi}{a}\right)\, d\xi \\
&\quad + \sinh\left(\frac{n\pi y}{a}\right)\int_y^b q_n(\xi)\sinh\left(\frac{n\pi(b-\xi)}{a}\right)\, d\xi \\
&= \int_0^b G(y,\xi)q_n(\xi)d\xi, \qquad (11)
\end{aligned}
$$

where

$$
G(y,\xi) = \begin{cases} \sinh\frac{n\pi(b-y)}{a}\sinh\left(\frac{n\pi\xi}{a}\right), & 0 \le \xi < y, \\ \sinh\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi(b-\xi)}{a}\right), & y < \xi \le b. \end{cases} \qquad (12)
$$

Thus we can get solution u_1 of Poisson's equation (with homogenous boundary conditions) using the x-dependent related one-dimensional homogeneous eigenfunctions:

$$u_1(x,y) = \sum_{n=1}^{\infty} \left(\int_0^b G(y,\xi) q_n(\xi) d\xi \right) \sin\left(\frac{n\pi x}{a}\right).$$

Problems with nonhomogeneous boundary conditions (that is the whole problem) can be solved by solving Laplace's equation with non-homogeneous boundary conditions (for u_2).

It may be noted that in this case, the distributed source $Q(x,y)$ is considered along $y = \xi$ (concentrated line source).

The problem can be reworked by considering the distributed source $Q(x,y)$ as concentrated source along $x = \zeta$ too.

For this case, the eigenfunctions $\sin\left(\frac{n\pi y}{b}\right)$ can be used to write the solution as

$$\begin{aligned} A_n(x) &= \sinh\left(\frac{n\pi(a-x)}{b}\right)\int_0^x q_n(\zeta)\sinh\left(\frac{n\pi\zeta}{b}\right)\,d\zeta \\ &\quad + \sinh\left(\frac{n\pi x}{b}\right)\int_x^a q_n(\zeta)\sinh\left(\frac{n\pi(a-\zeta)}{b}\right)\,d\zeta \\ &= \int_0^a G(x,\zeta)q_n(\zeta)d\zeta, \end{aligned} \tag{13}$$

where

$$G(x,\zeta) = \begin{cases} \sinh\frac{n\pi(a-x)}{b}\sinh\left(\frac{n\pi\zeta}{b}\right), & 0 \le \zeta < x, \\ \sinh\left(\frac{n\pi x}{b}\right)\sinh\left(\frac{n\pi(a-\zeta)}{b}\right), & x < \zeta \le a. \end{cases} \tag{14}$$

Thus we can get solution u_1 of Poisson's equation (with homogenous boundary conditions) using the x-dependent related one-dimensional homogeneous eigenfunctions:

$$u_1(x,y) = \sum_{n=1}^{\infty} \left(\int_0^a G(x,\zeta) q_n(\zeta) d\zeta \right) \sin\left(\frac{n\pi y}{b}\right).$$

Flashback 1

Solutions to linear partial differential equations are nonzero due to initial conditions, non-homogeneous boundary conditions and forcing terms.

Flashback 2

If the partial differential equation is homogeneous and there is a set of homogeneous boundary conditions, then we usually attempt to solve the problem by the method of separation of variables.

Remark 1

When we consider problems without initial conditions (ordinary differential equations, Laplace's equation with sources), we can show that there is one auxiliary function for each problem, called the Green's function, which can be used to describe the influence of both nonhomogeneous boundary conditions and forcing terms.

Remark 2

It is to be noted that time-dependent problems such as the ones governed by heat equation (parabolic equations) and wave equation (hyperbolic equations) are more difficult to tackle by this method.

Remark 3

In that sense, Green's functions for elliptic equations assumes more significance.

Remark 4

In other works, Green's functions can be suitably considered for steady state problems (BVP), but NOT for evolution problems (IVP or IBVP).