MA542, Differential Equations, Jan-May 2022, Differential Equations: PDEs in different coordinate systems Maximum principle for heat conduction equation

Lecture 43

26/04/2022

Steady-state heat conduction in a circular cylinder

Consider a right circular cylinder of radius a and height L having

(a) its convex surface and base in the xy-plane at temperature 0^0C,

(b) the top end $z = L$ is kept at temperature $f(r)^0$C.

To find the steady-state temperature at any point of the cylinder.

The governing equation for this problem will be Laplace's equation in r, θ, z.

$$\nabla^2 u(r,\theta,z) \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

For simplicity, we will consider *radially symmetric solution* for the Laplace's equation.

Radially symmetric solution means that $u(r,\theta,z) = u(r,z)$ that is the solution doesn't depend on the polar angle θ.

In other sense, solutions are symmetric under rotation.

Steady-state heat conduction in a circular cylinder

But assuming that the cylinder is symmetrical about its axis, Laplace's equation takes the form:

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0,\ 0 < r \le a,\ 0 \le z \le L. \tag{1}$$

The boundary conditions are:

$$\begin{aligned}
\text{(on the curved portion)}\ u(a,z) &= 0, 0 \le z \le L, && \text{(2a)}\\
\text{(on the bottom)}\ u(r,0) &= 0, 0 < r \le a, && \text{(2b)}\\
\text{(on the top)}\ u(r,L) &= f(r), 0 < r \le a. && \text{(2c)}
\end{aligned}$$

Assume a solution in the form

$$u(r,z) = R(r)Z(z).$$

Applying it to the governing equation (1):

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = 0.$$

Steady-state heat conduction in a circular cylinder

By separating the variables:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = k.$$

Observing that only the negative value of the separation constant will give rise to nontrivial solutions,

we get the following ODEs by considering $k = -\lambda^2$:

$$Z'' - \lambda^2 Z = 0, \tag{3}$$

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0, \tag{4}$$

The solutions of the above equations are, respectively, given by

$$Z(z) = A\sinh\lambda z + B\cosh\lambda z, \tag{5}$$

$$R(r) = CJ_0(\lambda r) + DY_0(\lambda r). \tag{6}$$

Steady-state heat conduction in a circular cylinder

The solution $u(r,z)$:

$$u(r,z) = (A\sinh\lambda z + B\cosh\lambda z)(CJ_0(\lambda r) + DY_0(\lambda r)) \tag{7}$$

We are looking for a bounded solution in $0 \le r \le a$,

we must take $D = 0$ since $Y_0 \to -\infty$ as $r \to 0$.

Equation (7) can be written as

$$u(r,z) = J_0(\lambda r)(A\sinh\lambda z + B\cosh\lambda z). \tag{8}$$

Now applying the boundary condition (2b), we get $B = 0$

implying

$$u(r,z) = AJ_0(\lambda r)\sinh\lambda z.$$

Steady-state heat conduction in a circular cylinder

Now applying the boundary condition (2a), we get $0 = AJ_0(\lambda a)\sinh\lambda z$

implying

$$J_0(\lambda a) = 0.$$

Hence

$$\lambda_n a = \nu_n,$$

where ν_n are the zeros of J_0.

The eigenvalues are given by

$$\lambda_n = \frac{\nu_n}{a}. \tag{9}$$

$$u_n(r,z) = A_n J_0\left(\frac{\nu_n}{a}r\right)\sinh\left(\frac{\nu_n}{a}z\right).$$

Steady-state heat conduction in a circular cylinder

By superposing all the solutions,

$$u(r,z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\nu_n}{a} r\right) \sinh\left(\frac{\nu_n}{a} z\right). \tag{10}$$

The coefficient A_n can be obtained by using the boundary condition (2c):

$$f(r) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\nu_n}{a} r\right) \sinh\left(\frac{\nu_n}{a} L\right), \tag{11}$$

giving us

$$A_n = \frac{\int_0^a r f(r) J_0\left(\frac{z_n}{a} r\right)\, dr}{\sinh\left(\frac{\nu_n}{a} L\right) \int_0^a r \left(J_0\left(\frac{z_n}{a} r\right)\right)^2\, dr}. \tag{12}$$

Cooling of a sphere

The form of Laplacian in spherical coordinates (r, θ, ϕ):

$$\nabla^2 u(r,\theta,\phi) = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{cot\theta}{r^2}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}.$$

Consider a classical problem in three dimensions:

the cooling of a sphere.

The assumed symmetries in the problem will allow us

to reduce the dimension of the problem to one spatial dimension and time only although it is a three-dimensional transient problem.

The problem is:

Given a sphere whose initial temperature depends only on the distance from the centre (e.g., a constant initial temperature) and whose boundary is kept at a constant temperature,

To predict the temperature at any point inside the sphere at a later time.

Cooling of a sphere

Consider a sphere of radius a whose initial temperature is T_0, a constant.

We assume that the boundary is held at zero degrees for all time $t > 0$.

If u is the temperature,

then in general u will depend on three spatial coordinates and time.

But a little of bit of careful observation will alow us to accept that

the temperature change can take place radially only.

Ultimately $u = u(r, t)$.

Cooling of a sphere

IBVP to be solved:

$$u_t = \alpha(u_{rr} + \frac{2}{r}u_r),\ 0 \le r \le a,\ t > 0, \tag{13}$$
$$u(a,t) = 0,\ t > 0, \tag{14}$$
$$u(r,0) = T_0,\ 0 \le r \le a. \tag{15}$$

Observe that there is an implied implicit boundary condition at $r = 0$:

namely that the temperature should remain bounded at all points (including $r = 0$) on the sphere.

Now assume a solution of the above IBVP in the form $u(r,t) = R(r)T(t)$

and apply separation of variables technique.

Cooling of a sphere

It gives us

$$\frac{T'}{\alpha T} = \left(\frac{R''}{R} + \frac{2}{r}\frac{R'}{R}\right) = k, \tag{16}$$

giving rise to the ODEs

$$\begin{aligned} R'' + \frac{2}{r}R' - kR &= 0, && (17) \\ T' + \alpha k T &= 0. && (18) \end{aligned}$$

The solution of (18) can be easily written as

$$T(t) = Ce^{-\alpha k t}. \tag{19}$$

To solve equation (17),

we introduce a new function $\Upsilon(r)$ defined by

$$\Upsilon(r) = rR(r).$$

Subsequently, (17) becomes

$$\Upsilon^{''}(r) - k\Upsilon(r) = 0. \tag{20}$$

Cooling of a sphere

If $k < 0$, say, $k = -\lambda^2$ (this is the only feasible case), then we have

$$\Upsilon(r) = A \sin \lambda r + B \cos \lambda r..$$

It gives $R(r)$ as

$$R(r) = \frac{1}{r} \left(A \sin \lambda \, r + B \cos \lambda \, r \right). \tag{21}$$

Since $(\cos \lambda r)/r$ is unbounded at $r = 0$, we must have $B = 0$ in order to have a bounded solution when $r \to 0$.

Hence, we write the solution as

$$u(r, t) = A \, \frac{\sin \lambda r}{r} \, e^{-\alpha \lambda^2 \, t}. \tag{22}$$

Cooling of a sphere

Applying the boundary condition (14), $\sin \lambda a = 0$.

It gives us the eigenvalues

$$\lambda_n = \frac{n\pi}{a}.$$

Therefore,

$$u_n(r,t) = A_n \, \frac{\sin\left(\frac{n\pi r}{a}\right)}{r} \, e^{-\alpha \frac{n^2\pi^2}{a^2}\, t}.$$

The complete solution can be written as

$$\begin{aligned} u(r,t) &= \sum_{n=1}^{\infty} u_n(r,t) \\ &= \sum_{n=1}^{\infty} A_n \, \frac{\sin\left(\frac{n\pi r}{a}\right)}{r} \, e^{-\alpha \frac{n^2\pi^2}{a^2} t}. \end{aligned} \tag{23}$$

Cooling of a sphere

Use the initial condition (15) to find A_n:

$$T_0 = \sum_{n=1}^{\infty} A_n \, \frac{\sin\left(\frac{n\pi r}{a}\right)}{r}.$$

It gives us A_n as

$$A_n = \frac{2T_0}{a} \int_0^a r \sin\left(\frac{n\pi r}{a}\right) \, dr. \tag{24}$$

It will ultimately, upon integration, give A_n as

$$A_n = (-1)^{n+1} \frac{a^2}{n\pi}. \tag{25}$$

Therefore, the solution can be finally written as

$$u(r,t) = \frac{2T_0 a^2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\sin\left(\frac{n\pi r}{a}\right)}{r} \, e^{-\alpha \frac{n^2\pi^2}{a^2} t}. \tag{26}$$

Cooling of a sphere

If the constant initial condition is replaced by some function of r, say $u(r,0) = f(r)$,

then also the solution is given by (23).

But the coefficient A_n is then is given by

$$A_n = \frac{2}{a}\int_0^a r f(r) \sin\frac{n\pi r}{a}\, dr. \tag{27}$$

In a similar manner,

the IBVP can be formulated with a non-zero constant temperature on the boundary and in that case, the IBVP has to be split into two problems: one IBVP with zero BC and another BVP which takes care of the non-homogeneous term in BC.

The Maximum and Minimum Principle for Heat Equation

To begin with, we shall first prove the maximum principle for the inhomogeneous heat equation ($F \neq 0$).

Theorem (The maximum principle for inhomogeneous equation)

Let $R: 0 \leq x \leq L,\ 0 \leq t \leq T$ be a closed region and let $u(x,t)$ be a solution of

$$u_t - \alpha u_{xx} = F(x,t) \ \text{ for } (x,t) \in R, \tag{28}$$

which is continuous on R.

- *If $F < 0$ in R, then $u(x,t)$ attains its maximum values on $t = 0$, $x = 0$ or $x = L$ (not in the interior of the region or at $t = T$).*
- *If $F > 0$ in R, then $u(x,t)$ attains its minimum values on $t = 0$, $x = 0$ or $x = L$ (not in the interior of the region or at $t = T$).*

Proof. We shall show that, if a maximum or minimum occurs at an interior point $0 < x_0 < L$ and $0 < t_0 \leq T$, then we will arrive at a contradiction.

The Maximum and Minimum Principle for Heat Equation (Contd.)

Case I: Let $F < 0$. Since $u(x,t)$ is continuous in a closed and bounded region in R, $u(x,t)$ must attain its maximum in R. Let (x_0, t_0) be the interior maximum point. Then, we must have

$$u_{xx}(x_0,t_0) \leq 0, \qquad u_t(x_0,t_0) \geq 0. \tag{29}$$

Since $u_x(x_0,t_0) = 0 = u_t(x_0,t_0)$, we have

$$u_t(x_0,t_0) = 0 \;\; \text{if} \;\; t_0 < T.$$

If $t_0 = T$, the point $(x_0,t_0) = (x_0,T)$ is on the boundary of R,

then we claim that

$$u_t(x_0,t_0) \geq 0$$

as u may be increasing at (x_0,t_0).

Substituting (29) in (28), we find that the left side of the equation (28) is non-negative while the right side is strictly negative.

This leads to a contradiction and hence, the maximum must be assumed on the initial line or on the boundary.

The Maximum and Minimum Principle for Heat Equation (Contd.)

Case II: Let $F > 0$.

Let there be an interior minimum point (x_0, t_0) in R such that

$$u_{xx}(x_0, t_0) \geq 0, \qquad u_t(x_0, t_0) \leq 0. \tag{30}$$

Note that the inequalities (30) are same as (29) with the signs reversed.

Again arguing as before, this leads to a contradiction, hence the minimum must be assumed on the initial line $t = 0$ or on the physical boundary.

Note:

When $F = 0$, i.e., for a homogeneous equation, the inequalities (29) at a maximum or (30) at a minimum do not lead to a contradiction when they are inserted into (28) as u_{xx} and u_t may both vanish at (x_0, t_0).

The Maximum and Minimum Principle for Heat Equation (Contd.)

Theorem (The maximum principle for homogeneous equation)

Let $R : 0 \leq x \leq L$ and $0 \leq t \leq T$. Let $u(x,t)$ be a solution of

$$u_t = \alpha u_{xx}, \tag{31}$$

which is continuous in the closed region R. The maximum and minimum values of $u(x,t)$ are attained on the initial line $t = 0$ or at the points on the boundary $x = 0$ or $x = L$.

Proof.

Let us introduce the auxiliary function $v(x,t)$ as

$$v(x,t) = u(x,t) + \epsilon x^2, \tag{32}$$

where $\epsilon > 0$ is a constant and u satisfies (31).

Note that $v(x,t)$ is continuous in R and hence it has a maximum at some point (x_1, t_1) in the region R.

The Maximum and Minimum Principle for Heat Equation (Contd.)

Assume that (x_1, t_1) is an interior point with $0 < x_1 < L$ and $0 < t_1 \leq T$.

Then we find that

$$v_t(x_1, t_1) \geq 0, \quad v_{xx}(x_1, t_1) \leq 0. \tag{33}$$

Since u satisfies (31), we have

$$v_t - \alpha v_{xx} = u_t - \alpha u_{xx} - 2\alpha\epsilon = -2\alpha\epsilon < 0. \tag{34}$$

Substituting (33) into (31) and using (34) now leads to

$$0 \leq v_t - \alpha v_{xx} < 0,$$

which is a contradiction since the left side is non-negative and the right side is strictly negative. Therefore, $v(x, t)$ assumes its maximum on the initial line or on the boundary since v satisfies (28) with $F = -2\alpha\epsilon < 0$.

The Maximum and Minimum Principle for Heat Equation (Contd.)

Let

$$M = \max\{u(x,t)\} \text{ on } t = 0, x = 0, \text{ and } x = L,$$

i.e., M is the maximum value of u on the initial line and boundary lines.

Then

$$v(x,t) = u(x,t) + \epsilon x^2 \le M + \epsilon L^2, \text{ for } 0 \le x \le L,\ 0 \le t \le T. \tag{35}$$

Since v has its maximum on $t = 0$, $x = 0$, or $x = L$, we obtain

$$u(x,t) = v(x,t) - \epsilon x^2 \le v(x,t) \le M + \epsilon L^2. \tag{36}$$

Since ϵ is arbitrary, letting $\epsilon \to 0$, we conclude that

$$u(x,t) \le M \text{ for all } (x,t) \in R, \tag{37}$$

and this completes the proof.

The Maximum and Minimum Principle for Heat Equation (Contd.)

Remarks.

1. The minimum principle for the heat equation can be obtained by replacing the function $u(x,t)$ by $-u(x,t)$, where $u(x,t)$ is a solution of (31). Clearly, $-u$ is also a solution of (31) and the maximum values of $-u$ correspond to the minimum values of u. Since $-u$ satisfies the maximum principle, we conclude that u assumes its minimum values on the initial line or on the boundary lines.

2. In geometrical sense, the maximum principle states that, if a solution of the problem (31) is graphed in the xtu-space, then the surface $u = u(x,t)$ achieves its maximum height above one of the three sides $x = 0, x = L, t = 0$ of the rectangle $0 \leq x \leq L$, $0 \leq t \leq T$.

3. From a physical perspective, the maximum principle states that the temperature, at any point x on the rod at any time t $(0 \leq t \leq T)$, is less than the maximum of the initial temperature distribution or the maximum of the temperatures prescribed at the ends during the time interval $[0,T]$.

Uniqueness and continuous dependence

As a consequence of the maximum principle, we can show that the heat flow problem has a unique solution and depends continuously on the given initial and boundary data.

Theorem (Uniqueness result)

Let $u_1(x,t)$ and $u_2(x,t)$ be solutions of the following problem

$$\begin{aligned} \textit{PDE:} \quad & u_t = \alpha u_{xx}, \quad 0 < x < L, \;\; t > 0, \\ \textit{BC:} \quad & u(0,t) = g(t), \quad u(L,t) = h(t), \\ \textit{IC:} \quad & u(x,0) = f(x), \end{aligned} \tag{38}$$

where $f(x)$, $g(t)$ and $h(t)$ are given functions. Then $u_1(x,t) = u_2(x,t)$, for all $0 \leq x \leq L$ and $t \geq 0$.

Uniqueness and continuous dependence (Contd.)

Proof.

Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of (38). Set $w(x,t) = u_1(x,t) - u_2(x,t)$.

Then w satisfies

$$\begin{aligned} & w_t = \alpha w_{xx}, \quad 0 < x < L, \;\; t > 0, \\ & w(0,t) = 0, \quad w(L,t) = 0, \\ & w(x,0) = 0. \end{aligned}$$

By the maximum principle, we must have

$$w(x,t) \leq 0 \Longrightarrow u_1(x,t) \leq u_2(x,t), \;\; \text{for all } 0 \leq x \leq L, \;\; t \geq 0.$$

A similar argument with $\bar{w} = u_2 - u_1$ yields

$$u_2(x,t) \leq u_1(x,t) \;\; \text{for all } 0 \leq x \leq L, \;\; t \geq 0.$$

Therefore, we have

$$u_1(x,t) = u_2(x,t) \;\; \text{for all } 0 \leq x \leq L, \;\; t \geq 0,$$

and this completes the proof.

Continuous dependence on the given IC and BC

Theorem (Continuous dependence on the given IC and BC)

Let $u_1(x,t)$ *and* $u_2(x,t)$*, respectively, be solutions of the problems*

$$\begin{aligned} u_t = \alpha u_{xx}; &\qquad u_t = \alpha u_{xx} \\ u(0,t) = g_1(t) \;\; u(L,t) = h_1(t); &\qquad u(0,t) = g_2(t) \;\; u(L,t) = h_2(t) \\ u(x,0) = f_1(x); &\qquad u(x,0) = f_2(x), \end{aligned} \tag{39}$$

in the region $0 \leq x \leq L$, $0 \geq t \leq T$. *If*

$$|f_1(x) - f_2(x)| \leq \epsilon \;\text{ for all } x,\; 0 \leq x \leq L,$$

and

$$|g_1(t) - g_2(t)| \leq \epsilon \;\text{ and }\; |h_1(t) - h_2(t)| \leq \epsilon \;\text{ for all } t,\; 0 \leq t \leq T,$$

for some $\epsilon \geq 0$*, then we have*

$$|u_1(x,t) - u_2(x,t)| \leq \epsilon \;\text{ for all } x \text{ and } t, \text{where } \; 0 \leq x \leq L,\; 0 \leq t \leq T.$$

Continuous dependence on the given IC and BC (Contd.)

Proof.

Let $v(x,t) = u_1(x,t) - u_2(x,t)$. Then $v_t = \alpha v_{xx}$ and we obtain

$$
\begin{aligned}
|v(x,0)| &= |f_1(x) - f_2(x)| \leq \epsilon, \;\; 0 \leq x \leq L, \\
|v(0,t)| &= |g_1(t) - g_2(t)| \leq \epsilon, \;\; 0 \leq t \leq T, \\
|v(l,t)| &= |h_1(t) - h_2(t)| \leq \epsilon, \;\; 0 \leq t \leq T.
\end{aligned}
$$

Note that the maximum of v on $t = 0$ $(0 \leq x \leq L)$ and $x = 0$ and $x = l$ $(0 \leq t \leq T)$ is not greater than ϵ. The minimum of v on these boundary lines is not less than $-\epsilon$.

Hence, the maximum/minimum principle yields

$$-\epsilon \leq v(x,t) \leq \epsilon \implies |u_1(x,t) - u_2(x,t)| = |v(x,t)| \leq \epsilon.$$

Continuous dependence on the given IC and BC (Contd.)

Remarks.

- We observe that when $\epsilon = 0$, the problems in (39) are identical. We conclude that $|u_1(x,t) - u_2(x,t)| \leq 0$ (i.e., $u_1 = u_2$). This proves the uniqueness result.
- Suppose a certain initial/boundary value problem has a unique solution. Then a small change in the initial and/or boundary conditions yields a small change in the solutions.

For the inhomogeneous equation (28), we have seen that the maximum or minimum values must be attained either on the initial line or the boundary lines and that they cannot be assumed in the interior.

The following result is known as a strong maximum or minimum principle.

Theorem (Strong maximum principle)

Let $u(x,t)$ be a solution of the heat equation in the rectangle $R:\ 0 \leq x \leq L,\ 0 \leq t \leq T$. If $u(x,t)$ achieves its maximum at (x^*, T), where $0 < x^* < L$, then u must be constant in R.

Time-Independent Non-homogeneous BC

We now turn to the situation where the BC are not both homogeneous, but are independent of time variable t.

The method of solution consists of the following steps:

- Step 1: Find a particular solution of the PDE and BC.
- Step 2: Find the solution of a related problem with homogeneous BC. Then, add this solution to that particular solution obtained in Step 1.

The procedure is illustrated in the following example:

$$\text{PDE:} \qquad u_t = \alpha u_{xx}, \quad 0 \le x \le L,\ t > 0, \tag{40}$$

$$\text{BC:} \qquad u(0,t) = a,\ \ u(L,t) = b, \quad t > 0, \tag{41}$$

$$\text{IC:} \qquad u(x,0) = f(x), \quad 0 \le x \le L, \tag{42}$$

where a and b are arbitrary constants and $f(x)$ is a given function.

Time-Independent Non-homogeneous BC (Contd.)

Solution.

Seek a particular solution $u_p(x)$ of the form $u_p(x) = cx + d$, where c and d are to be chosen so that the BC are satisfied:

$$\begin{aligned} & a = u_p(0) = c \cdot 0 + d = d, \\ & b = u_p(L) = cL + d = cL + a. \\ \implies \quad & d = a \quad \text{and} \quad c = (b-a)/L. \end{aligned}$$

Thus

$$u_p(x) = (b-a)x/L + a$$

solves the PDE with the BC satisfied.

Consider the related homogeneous problem (i.e., with homogeneous PDE and BC)

$$\begin{aligned} \text{PDE:} \quad & v_t = \alpha v_{xx}, \quad 0 \le x \le L,\ t > 0, \\ \text{BC:} \quad & v(0,t) = 0,\ \ v(L,t) = 0, \quad t > 0, \\ \text{IC:} \quad & v(x,0) = f(x) - u_p(x), \quad 0 \le x \le L. \end{aligned} \tag{43}$$

Time-Independent Non-homogeneous BC (Contd.)

If $f(x) - u_p(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$, then its solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/l)^2 \alpha t} \sin(n\pi x/L).$$

Now, set $u(x,t) = v(x,t) + u_p(x)$.

Then it is easy to verify that $u(x,t)$ solves (40). Indeed, $u(x,t)$ solves (40) by the superposition principle. Further, we have

$$\begin{aligned}
\text{BC:} \quad & u(0,t) = v(0,t) + u_p(0) = 0 + a = a, \\
& u(L,t) = v(L,t) + u_p(L) = 0 + b = b, \\
\text{IC:} \quad & u(x,0) = v(x,0) + u_p(x) = f(x) - u_p(x) + u_p(x) = f(x).
\end{aligned}$$

Time-Independent Non-homogeneous BC (Contd.)

Remarks:

- It is necessary to subtract $u_p(x)$ from $f(x)$ to form the initial condition for the related problem (43) so that the initial condition (42) is satisfied.
- Since any particular solution will do, for simplicity one should consider a particular solution of the form $cx + d$, and find the constants, using the BC. The reason is that the formula only applies to the BC of (41). For other BC, we obtain other particular solution. For example, If $u_x(0,t) = a,\ u(L,t) = b$, then $u_p(x) = a(x - L) + b$.

Time-Independent Non-homogeneous BC (Contd.)

Example

$$\text{PDE:} \quad u_t = 2u_{xx}, \quad 0 \le x \le 1,\ t > 0, \tag{44}$$

$$\text{BC:} \quad u_x(0,t) = 1,\ \ u(1,t) = -1, \quad t > 0, \tag{45}$$

$$\text{IC:} \quad u(x,0) = x + \cos^2(3\pi x/4) - 5/2. \tag{46}$$

Solution.

Take $u_p(x) = cx + d$. The first BC $u_x(0,t) = 1$ yields $c = 1$, while $u_p(1) = 1 + d$ yields $d = -2$ by the second BC. Thus, $u_p(x) = x - 2$.

The related homogeneous problem is

$$\begin{aligned} &v_t = 2v_{xx}, \quad 0 \le x \le 1,\ t > 0, \\ &v_x(0,t) = 0,\ \ v(1,t) = 0, \quad t > 0, \\ &v(x,0) = [x + \cos^2(3\pi x/4) - 5/2] - (x - 2) \\ &\qquad\quad = \frac{1}{2} + \frac{1}{2}\cos(3\pi x/2) - 5/2 + 2 = \frac{1}{2}\cos(3\pi x/2). \end{aligned}$$

Time-Independent Non-homogeneous BC (Contd.)

It is easy to obtain the solution of the related homogeneous problem as

$$v(x,t) = e^{-9\pi^2 t/2}[\frac{1}{2}\cos(3\pi x/2)].$$

From the above examples, we notice that the particular solution is time independent, or in steady-state.

Note:

Any steady-state solution of the heat equation $u_t = \alpha u_{xx}$ is of the form $cx + d$.

The solutions $u(x,t)$ are sums of a steady-state particular solution of the PDE and BC and the solution $v(x,t)$ of the related homogeneous problem which is transient in the sense that $v(x,t) \to 0$ as $t \to \infty$.

Time-Independent Non-homogeneous BC (Contd.)

Thus

$$u(x,t) \;=\; \underbrace{u_p(x)}_{\text{(steady-state solution)}} \;+\; \underbrace{v(x,t)}_{\text{(transient solution)}} \;\to\; u_p(x), \quad \text{as } t \to \infty.$$

That is, the solution u approaches the steady-state solution as $t \to \infty$.

However, for some types of BC, there are no steady-state particular solutions.

Example

Consider the problem

$$\text{PDE:} \qquad u_t = \alpha u_{xx}, \quad 0 \le x \le L, \; t > 0, \tag{47}$$

$$\text{BC:} \qquad u_x(0,t) = a, \;\; u_x(l,t) = b, \tag{48}$$

$$\text{IC:} \qquad u(x,0) = f(x), \tag{49}$$

where a and b are constants, and $f(x)$ is a given function.

Time-Independent Non-homogeneous BC (Contd.)

Solution.

Let $u_p(x) = cx + d$. Then, using BC, we obtain $c = a$ and $c = b$, which is impossible unless $a = b$.

NOTE:

Observe that the boundary conditions state that heat is being drained out of the end $x = 0$ at a rate $u_x(0, t) = a$ and heat is flowing into the end $x = l$ at a rate $u_x(l, t) = b$. If $b > a$, then the heat energy is being added to the rod at a constant rate. If $b < a$, the rod loses heat at a constant rate. Thus, we cannot expect a steady-state solution of the PDE and BC, unless $a = b$.