MA542, Jan-May 2022, Differential Equations

Laplace's equation

Lecture 41

22/04/2022

Steady state heat conduction: Laplace's equation

Laplace's equation in two or three dimensions

usually arises in two types of physical problems

1. Steady state heat conduction.
2. As equation of continuity for incompressible potential flow.

However, here we will emphasize only

on the first type

Steady state solution here means

the solution for large time.

Laplace's equation: Physical Meaning

Laplace's equation in two dimensions and three dimensions, are , respectively, given by

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & (1) \\ u_{xx} + u_{yy} + u_{zz} &= 0. & (2) \end{aligned}$$

The above equations can be obtained from the two-dimensional and three-dimensional transient heat conduction equations when u does not depend on t.

Hence Laplace's equation models

steady heat flow in a region where the temperature is fixed on the boundary.

Let us take up Laplace's equation in two-dimensions and examine

what its tells us from a physical point of view.

Laplace's equation: Physical Meaning

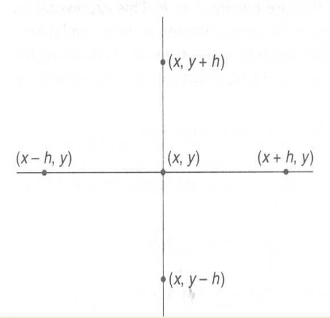

Figure : Laplace's equation in a rectangular region

Let (x, y) be some point in the region where heat is flowing.

Let h be some small distance.

Laplace's equation: Physical Meaning

By Taylor's theorem for two variables,

$$u(x-h,y) = u(x,y) - hu_x(x,y) + \frac{1}{2}h^2u_{xx}(x,y) - \frac{1}{6}h^3u_{xxx}(x,y) + O(h^4), \tag{3}$$

where

$O(h^4)$ denotes the remaining terms (the error term), which are of at least power 4 in h.

Similarly

$$u(x+h,y) = u(x,y) + hu_x(x,y) + \frac{1}{2}h^2u_{xx}(x,y) + \frac{1}{6}h^3u_{xxx}(x,y) + O(h^4). \tag{4}$$

Adding equations (3) and (4) and solving for u_{xx}:

$$u_{xx}(x,y) = \frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2} + O(h^2), \tag{5}$$

It is a difference-quotient approximation to the second partial derivative u_{xx} at (x,y), and the error is proportional to h^2.

Laplace's equation: Physical Meaning

Now incrementing y instead of x, the approximation for u_{yy}:

$$u_{yy}(x,y) = \frac{u(x,y-h) - 2u(x,y) + u(x,y+h)}{h^2} + O(h^2), \tag{6}$$

(5) and (6) $\Rightarrow$

Laplace's equation at (x,y) can be approximated by the equation

$$\frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2} + \frac{u(x,y-h) - 2u(x,y) + u(x,y+h)}{h^2} + O(h^2) = 0. \tag{7}$$

Solving for $u(x,y)$ gives, upon neglecting the small-order terms,

$$u(x,y) \approx \frac{1}{4}[u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h)]. \tag{8}$$

Equation (8) states, physically,

that the temperature at a point (x,y) is approximately (since we neglected small terms) the average of the temperatures at four nearby points $(x-h,y), (x+h,y), (x,y-h), (x,y+h)$.

Laplace's equation: Physical Meaning

Observe that the temperature at (x, y)

cannot exceed the temperatures at all the neighbouring points;

so a maximum cannot occur at (x, y).

Similarly, the temperature at (x, y) cannot

be less than the temperatures at all the neighbouring points;

so a minimum also cannot occur at (x, y).

This important physical interpretation can be extended to a circle.

Laplace's equation: Physical Meaning

If $u = u(x, y)$ satisfies Laplace's equation at a point $P_0 : (x_0, y_0)$ in a region,

then the temperature at P_0 is the average value of the temperature on any circle $C_R(P_0)$ of radius R centered at P_0 lying in the region:

$$u(x_0, y_0) = \frac{1}{2\pi R} \int_{C_R(P_0)} u(x, y) ds.$$

The integral here is a line integral over the curve $C_R(P_0)$.

This result can be generalized to three dimensions:

The steady-state temperature at a point is the average of the temperatures over the surface of any sphere centred at that point.

Laplace's equation: Physical Meaning

If the value of u at a point is the average of the values on any circle about that point,

then the value of u at that point cannot exceed every value of u on any given circle.

Intuitively,

this seems to imply that a function u satisfying Laplace's equation in a given domain cannot have a local maximum at a point inside that domain;

the maximum must therefore occur on the boundary of the domain.

Indeed, this is true, and

the result is called the maximum principle.

Maximum Principle

Theorem: Let $u(x, y)$ satisfy Laplace's equation in D, an open, bounded, connected region in the plane; and let u be continuous on the closed domain $D \cup \partial D$ consisting of D and its boundary. If u is not a constant function, then the maximum and minimum values of u are attained on the boundary of D and nowhere inside D.

. $\square$

This is called maximum principle theorem for Laplace's equation.

Steady state heat conduction in two dimensions

We consider steady state heat conduction

in a two-dimensional rectangular region.

To be specific,

consider the equilibrium temperature inside a rectangle $0 \le x \le a, \ 0 \le y \le b$.

Here

the temperature is a prescribed function of position on the boundary.

In general the Dirichlet BVP will be like

$$u_{xx} + u_{yy} = 0, \ \ 0 \le x \le a, \ 0 \le y \le b$$
$$u(0, y) = g_1(y), \ u(a, y) = g_2(y), \ \ 0 \le y \le b$$
$$u(x, 0) = f_1(x), \ u(x, b) = f_2(x), \ \ 0 \le x \le a.$$

Steady state heat conduction in two dimensions

where

$f_1(x), f_2(x), g_1(y), g_2(y)$ are given functions.

Though the equation is linear and homogenous,

the BCs are not homogenous.

Hence

the BVP is needed to be split into four BVPs with each containing one non-homogenous BC.

Take

$$u = u_1 + u_2 + u_3 + u_4, \;\; 0 \leq x \leq a, \; 0 \leq y \leq b.$$

Steady state heat conduction in two dimensions

BVP I and BVP II:

$$
\begin{aligned}
u_{1,xx} + u_{1,yy} &= 0; & u_{2,xx} + u_{2,yy} &= 0,\ \ 0 \le x \le a,\ 0 \le y \le b;\\
u_1(0,y) &= 0,\ 0 \le y \le b; & u_2(0,y) &= 0,\ 0 \le y \le b;\\
u_1(a,y) &= 0,\ 0 \le y \le b; & u_2(a,y) &= 0,\ 0 \le y \le b;\\
u_1(x,0) &= f_1(x),\ 0 \le x \le a; & u_2(x,0) &= 0,\ 0 \le x \le a;\\
u_1(x,b) &= 0,\ 0 \le x \le a; & u_2(x,b) &= f_2(x),\ 0 \le x \le a;
\end{aligned}
$$

BVP III and BVP IV:

$$
\begin{aligned}
u_{3,xx} + u_{3,yy} &= 0; & u_{4,xx} + u_{4,yy} &= 0,\ \ 0 \le x \le a,\ 0 \le y \le b;\\
u_3(0,y) &= g_1(y),\ 0 \le y \le b; & u_4(0,y) &= 0,\ 0 \le y \le b;\\
u_3(a,y) &= 0,\ 0 \le y \le b; & u_4(a,y) &= g_2(y),\ 0 \le y \le b;\\
u_3(x,0) &= 0,\ 0 \le x \le a; & u_4(x,0) &= 0,\ 0 \le x \le a;\\
u_3(x,b) &= 0,\ 0 \le x \le a; & u_4(x,b) &= 0,\ 0 \le x \le a.
\end{aligned}
$$

We will consider only one of them.......take $u_1 = u$ for convenience

Steady state heat conduction in two dimensions

Consider the steady state heat conduction in a rectangular region $0 \leq x \leq a,\ 0 \leq y \leq b$

where three boundaries along $x = 0, x = a, y = b$ are kept at 0^0C

while

the temperature along the boundary $y = 0$ is $f(x)$.

$\boxed{\text{To find}}$ the temperature at any point (x, y).

Steady state heat conduction in two dimensions

BVP will consist of the following:

The governing equation is two-dimensional Laplace's equation:

$$u_{xx} + u_{yy} = 0, \;\; 0 \le x \le a, \; 0 \le y \le b. \tag{9}$$

The boundary conditions are:

$$\begin{aligned} u(0,y) &= 0, \; 0 \le y \le b, && \text{(10a)} \\ u(a,y) &= 0, \; 0 \le y \le b, && \text{(10b)} \\ u(x,0) &= f(x), \; 0 \le x \le a, && \text{(10c)} \\ u(x,b) &= 0, \; 0 \le x \le a. && \text{(10d)} \end{aligned}$$

It being a pure BVP and the solution being a function of x and y,

obviously we will not have any initial conditions.

Hence

This problem is called a steady-state problem.

Steady state heat conduction in two dimensions

Assume a solution of the form:

$$u(x, y) = X(x)Y(y). \tag{11}$$

Using (11) in (9)

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

On separating the variables x and y,

$$\frac{X''}{X} = -\frac{Y''}{Y} = k(\text{say}).$$

Giving us

$$X'' - kX = 0, \tag{12}$$
$$Y'' + kY = 0. \tag{13}$$

The zero and positive values of k will not give rise to solutions conforming to the boundary conditions.

Steady state heat conduction in two dimensions

We consider only the negative values of k, say $-\lambda^2$, to write the equations (12) and (13) as

$$X'' + \lambda^2 X = 0, \quad (14)$$
$$Y'' - \lambda^2 Y = 0, \quad (15)$$

so that the solution $u(x, y)$ can be written as

$$u(x, y) = (A\cos\lambda x + B\sin\lambda x)(C\cosh\lambda y + D\sinh\lambda y). \quad (16)$$

Using boundary condition (10a)

$A = 0.$

Using boundary condition (10b),

$$\lambda_n = \frac{n\pi}{a},\ n = 1, 2, 3, \dots$$

$\Rightarrow$

$$u_n(x, y) = \sin\frac{n\pi x}{a}\left(A_n\cosh\frac{n\pi y}{a} + B_n\sinh\frac{n\pi y}{a}\right).$$

Steady state heat conduction in two dimensions

Using boundary condition (10d)

$$B_n = -\frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}}\, A_n$$

so that the solution $u(x,y)$ can be written as

$$\begin{aligned} u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) &= \sum_{n=1}^{\infty} A_n\, \sin\frac{n\pi x}{a}\left(\cosh\frac{n\pi y}{a} - \frac{\cosh\frac{n\pi b}{a}}{\sinh\frac{n\pi b}{a}}\sinh\frac{n\pi y}{a}\right) \\ &= \sum_{n=1}^{\infty} A_n\, \sin\frac{n\pi x}{a}\, \frac{\sinh\frac{n\pi(b-y)}{a}}{\sinh\frac{n\pi b}{a}} \qquad (17) \end{aligned}$$

Remaining boundary condition (10c) can be used to evaluate the coefficients A_n:

$$f(x) = \sum_{n=1}^{\infty} A_n\, \sin\frac{n\pi x}{a}.$$

A_n is obtained as

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\, dx. \qquad (18)$$

The solution to the BVP described by equations (9)-(10) is given by

(17) with A_n given by (18).

Steady state heat conduction in two dimensions

Similarly we can find the other solutions u_2, u_3 and u_4 and

write the total solution as $u = u_1 + u_2 + u_3 + u_4$.

This problem with Dirichlet conditions

along all boundaries is called a Dirichlet problem for a rectangle.

The problem with Neumann conditions

along all boundaries is called a Neumann problem for a rectangle.

This new problem can be solved by writing the boundary conditions as

$$u_x(0, y) = 0, \tag{19a}$$
$$u_x(a, y) = 0, \tag{19b}$$
$$u_y(x, 0) = f(x), \tag{19c}$$
$$u_y(x, b) = 0. \tag{19d}$$

TRY to solve it yourself.