MA542 January-May2022, Differential Equations 1D heat conduction equation

Lecture 40

21/04/2022

Heat conduction problem in a thin rod

Consider following heat conduction equation in a rod $(0, L)$

$$u_t = \alpha u_{xx}, \quad 0 < x < L, \quad t > 0. \tag{1}$$

The initial condition for $0 \leq x \leq L$:

$$u(x, 0) = f(x). \tag{2}$$

In practice, temperature $u(x, t)$ satisfies certain boundary conditions such as:

(a) Dirichlet Condition: $u(0, t) = u_0, \ u(L, t) = u_L, \ t > 0$

(b) Neumann Condition: $u_x(0, t) = \alpha_1, \ u_x(L, t) = \beta_1, \ t > 0,$

(c) Robin Condition: $u_x(0, t) + a_0 u(0, t) = \alpha_2, \ u_x(L, t) + a_L u(L, t) = \beta_2.$

Physically, u_x denotes heat flux. So, $u_x(0, t) = 0$ means

the left end of the rod is insulated, i.e., heat transfer through that point is zero.

Heat conduction problem with flux conditions

Consider the IBVP consisting of the following:

$$u_t = \alpha u_{xx}, \quad 0 < x < L, \quad t > 0, \tag{3}$$

The boundary conditions for all $t > 0$:

$$u_x(0,t) = 0, \tag{4a}$$
$$u(L,t) = 0. \tag{4b}$$

The initial condition for $0 \leq x \leq L$:

$$u(x,0) = \phi(x). \tag{5}$$

The boundary conditions (4) tell us that

the left end of the rod is insulated and the right end is kept at zero degrees.

Heat conduction problem with flux conditions

Using the separation of variable technique by assuming a solution of the form $u(x,t) = X(x)T(t)$, the PDE (3) is converted to the following ODEs:

$$\begin{aligned} X'' - kX &= 0, \\ T' - k\alpha T &= 0. \end{aligned}$$

Taking $k = -\lambda^2$, the above equations become

$$X'' + \lambda^2 X = 0, \tag{6}$$

$$T' + \lambda^2 \alpha T = 0. \tag{7}$$

Giving us the solutions:

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{8}$$

$$T(t) = Ce^{-\alpha\lambda^2 t}. \tag{9}$$

Heat conduction problem with flux conditions

The solution:

$$u(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)]Ce^{-\alpha\lambda^2 t}. \tag{10}$$

To use the boundary condition (4a), we have to differentiate (10) w.r.t. x to get

$$u_x = \lambda[-A\sin(\lambda x) + B\cos(\lambda x)]Ce^{-\alpha\lambda^2 t}. \tag{11}$$

Using boundary condition (4a)

$B = 0$ and hence

$$u(x,t) = A\cos(\lambda x)e^{-\alpha\lambda^2 t}. \tag{12}$$

The boundary condition (4b) will give the eigenvalues as

$$\lambda_n = \left(\frac{2n+1}{2}\right)\frac{\pi}{L},\ n = 0, 1, 2, 3, \ldots .$$

Heat conduction problem with flux conditions

The solution to the IBVP will be given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \,\text{exp}\left[-\alpha\left(\frac{2n+1}{2}\right)^2 \frac{\pi^2}{L^2} t\right] \cos\left(\frac{(2n+1)\pi x}{2L}\right), \tag{13}$$

where A_n can be obtained from IC (5):

$$\phi(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

as

$$A_n = \frac{2}{L}\int_0^L f(x)\ \cos\left(\frac{(2n+1)\pi x}{2L}\right)\ dx,\ n = 0, 1, 2, 3, \ldots \tag{14}$$

Note that here $n = 0$ also contributes to the solution.

In other words, we can say that the eigenvalue λ_0, corresponding to $n = 0$, also contributes.

Heat conduction problem with different flux conditions at ends

Note the difference in solution

when Dirichlet condition at both ends of the rod are changed to one Neumann (at $x = 0$) and one Dirichlet (at $x = L$) conditions

We may have two other combinations of pairs of conditions, viz.,

3. Dirichlet condition at $x = 0$

and Neumann condition at $x = L$

4. Neumann condition at both ends $x = 0$ and $x = L$.

In other words,

We can have four types of problems corresponding to a pair-wise end conditions.

Heat conduction problems with different conditions at ends

Problem I

Governing equation: $u_t = \alpha u_{xx}, \ \ 0 < x < L, \ t > 0,$

Boundary conditions: $u(0,t) = 0 = u(L,t)$, Initial Condition: $u(x,0) = \phi(x)$.

Solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \ e^{-\alpha \frac{n^2\pi^2}{L^2} \ t},$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Heat conduction problems with different conditions at ends

Problem II

Governing equation: $u_t = \alpha u_{xx}, \;\; 0 < x < L, \; t > 0,$

Boundary conditions: $u_x(0,t) = 0 = u(L,t)$, Initial Condition: $u(x,0) = \phi(x)$.

The solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \; \cos\left(\frac{(2n+1)\pi x}{2L}\right) \; \exp\left[-\alpha\left(\frac{2n+1}{2}\right)^2 \frac{\pi^2}{L^2} t\right],$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cos(\frac{(2n+1)\pi x}{2L})dx.$$

Heat conduction problems with different conditions at ends

Problem III

Governing equation: $u_t = \alpha u_{xx}, \;\; 0 < x < L, \; t > 0,$

Boundary conditions: $u(0,t) = 0 = u_x(L,t)$, Initial Condition: $u(x,0) = \phi(x)$.

Solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \;\sin\left(\frac{(2n+1)\pi x}{2L}\right)\; \exp\left[-\alpha\left(\frac{2n+1}{2}\right)^2 \frac{\pi^2}{L^2} t\right],$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right) dx.$$

Heat conduction problems with different conditions at ends

Problem IV

Governing equation $u_t = \alpha u_{xx},, \;\; 0 < x < L, \; t > 0,$

Boundary conditions: $u_x(0,t) = 0 = u_x(L,t)$, Initial Condition: $u(x,0) = \phi(x)$.

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \; e^{-\alpha \frac{n^2\pi^2}{L^2} \, t},$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\cos(\frac{n\pi x}{L})dx.$$

Heat conduction problems with radiation conditions at one end

Consider a problem related to radiation condition.

Take up the following IBVP:

Governing equation:

$$u_t = \alpha u_{xx},\ 0 < x < L,\ t > 0. \tag{15}$$

Boundary Conditions:

$$u(0,t) = 0, \tag{16a}$$

$$u_x(L,t) + hu(L,t) = 0,\ t > 0, \tag{16b}$$

where h is a constant.

Initial condition:

$$u(x,0) = \phi(x),\ 0 \leq x \leq L. \tag{17}$$

Heat conduction problems with radiation conditions at one end

Solution can be written as

$$u(x,t) = [A\sin(\lambda x) + B\cos(\lambda x)]Ce^{-\alpha\lambda^2 t}. \tag{18}$$

Using boundary condition (16a)

we get $B = 0$.

Using boundary condition (16b),

$$[Ah\sin(\lambda L) + \lambda A\cos(\lambda L)]Ce^{-\alpha\lambda^2 t} = 0$$

ultimately giving us a transcendental equation for λ as

$$\lambda/h = -\tan(\lambda L). \tag{19}$$

Equation (19) cannot be solved analytically

but the graphs of the functions λ/h and $-\tan\lambda L$ versus λ show that the equation has infinitely many positive solutions $\lambda_1, \lambda_2, \ldots$

Heat conduction problems with radiation conditions at one end

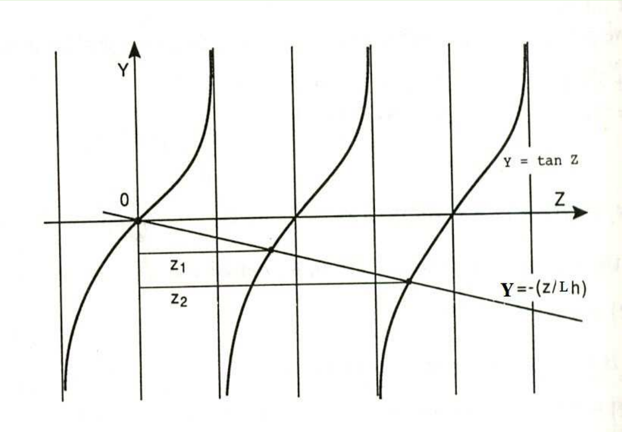

Figure : Graphical representation of the eigenvalues

Heat conduction problems with radiation conditions at one end

These values λ_n are the eigenvalues.

$$u_n(x,t) = A_n \sin(\lambda_n x) e^{-\alpha \lambda_n^2 t}. \tag{20}$$

Hence the solution:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)\ e^{-\alpha \lambda_n^2 t}, \tag{21}$$

where

λ_n are solutions of (19).

The initial condition will give us A_n as

$$A_n = \frac{2}{L} \int_0^L \phi(x) \sin(\lambda_n x)\ dx.$$

Non-Homogeneous Heat Equation: Duhamel's Principle

Find temperature $u(x,t)$ such that (when a source is included)

$$u_t - \alpha u_{xx} = f(x,t), \quad (x,t) \in (0,L) \times (0,\infty), \tag{22}$$

$$\text{with BCs } \; u(0,t) = 0, \;\; u(L,t) = 0, \; t > 0, \;\; \text{and} \tag{23}$$

$$\text{with IC } \; u(x,0) = 0, \;\; x \in [0,L]. \tag{24}$$

Then the solution u is given by

$$u(x,t) = \int_0^t v(x, t-\tau, \tau)\, d\tau. \tag{25}$$

What is v?

Duhamel's Principle (Contd.)

Here, v is the solution of the problem

$$v_t - \alpha v_{xx} = 0, \quad (x,t) \in (0,L) \times (0,\infty), \tag{26}$$

$$\text{with BCs } \; v(0,t) = 0, \;\; v(L,t) = 0, \; t > 0 \; \text{ and} \tag{27}$$

$$\text{with IC } \; v(x,0) = f(x,s), \quad s > 0. \tag{28}$$

for some real parameter $s > 0$.

Note that the solution v of the above problem depends on x, t and s.

Thus, $v = v(x,t,s)$. Accordingly, we can modify the BCs and IC as

$$\text{BCs: } \; v(0,t,s) = 0, \;\; v(L,t,s) = 0, \; t > 0, \; s > 0, \;\; \text{and}$$

$$\text{IC: } \; v(x,0,s) = f(x,s), \quad s > 0.$$

Leibniz Rule: Differentiation under the sign of integration

$$\frac{d}{dt}\Big(\int_{a(t)}^{b(t)} G(x,t)dx\Big) = \int_{a(t)}^{b(t)} G_t dx + G(b(t),t)b'(t) - G(a(t),t)a'(t). \quad (29)$$

Use Leibniz rule on

$$u(x,t) = \int_0^t v(x, t-\tau, \tau)d\tau$$

to have

$$\begin{aligned} u_t &= \int_0^t v_t(x, t-\tau, \tau)d\tau + v(x, t-t, t) \times \frac{dt}{dt} - v(x, t-0, 0) \times 0 \\ &= \int_0^t v_t(x, t-\tau, \tau)d\tau + v(x, 0, t) = \int_0^t v_t ds + f(x,t). \end{aligned}$$

Leibniz Rule: Differentiation under the sign of integration(contd.)

Similarly,

$$u_{xx}(x,t) = \int_0^t v_{xx} ds = \frac{1}{\alpha} \int_0^t v_t ds. \tag{30}$$

Combining above two equations, we have

$$u_t = \int_0^t v_t ds + f(x,t) = \alpha u_{xx} + f(x,t).$$

Further, u given by

$$u(x,t) = \int_0^t v(x, t-\tau, \tau) d\tau$$

satisfies both initial and boundary conditions.

Example

Solve

$$
\begin{aligned}
&u_t - u_{xx} = t\sin x,\ 0 < x < \pi,\\
&u(0,t) = 0,\ \ u(\pi,t) = 0,\ \ t > 0,\\
&u(x,0) = 0.
\end{aligned}
$$

Solution: We first solve the related problem for $v(x,t,s)$

$$
\begin{aligned}
&v_t - v_{xx} = 0,\ 0 < x < \pi,\\
&v(0,t,s) = 0,\ \ v(\pi,t,s) = 0,\ \ t > 0,\ s > 0,\\
&v(x,0,s) = F(x,s) = s\sin x.
\end{aligned}
$$

Example

From method of separation of variables, for fixed s, we obtain

$$v(x,t,s) = \sum_{n=1}^{\infty} B_n \sin nx \; e^{-n^2 t}, \tag{31}$$

with B_n as

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L F(x,s)\sin nx dx \\ &= \frac{2}{\pi}\int_0^{\pi} s\sin x \sin nx dx. \end{aligned}$$

Hence,

$$B_1 = s \quad \& \quad B_n = 0 \text{ for } n \neq 1. \tag{32}$$

Example

For any $s > 0$, we obtain

$$v(x,t,s) = se^{-t}\sin x. \tag{33}$$

Therefore

$$v(x,t-\tau,\tau) = \tau e^{-(t-\tau)}\sin x.$$

Hence, due to Duhamel's principle, $u(x,t)$ is given by

$$\begin{aligned} u(x,t) &= \int_0^t v(x,t-\tau,\tau)d\tau = \int_0^t \tau e^{-(t-\tau)}\sin x d\tau \\ &= e^{-t}\sin x \int_0^t \tau e^{\tau} d\tau. \end{aligned}$$