MA 542 Differential Equations
Lecture 4
(January 11, 2022)

By now, we realize that

there are four major types of first-order ordinary differential equations: separable, homogeneous, exact and linear.

There are some nonlinear differential equations which can be reduced to linear form by suitable change of dependent variable.

We will discuss three special types of nonlinear differential equations and their solution techniques:

1. Bernoulli's equation
2. Riccati equation
3. Equation of Clairaut

I. Bernoulli's equation

Bernoulli's equation has the following form:

$$y' + P(x)y = Q(x)y^n, \tag{1}$$

where n may take any value. If $n = 0$ or $n = 1$, this equation becomes linear.

For all other values of n, (1) is nonlinear. It can be written as

$$y^{-n}y' + P(x)y^{1-n} = Q(x). \tag{2}$$

Putting $z = y^{1-n}$:

$$z' = (1-n)y^{-n}y'.$$

Substituting in (2):

$$z' + (1-n)P(x)z = (1-n)Q(x), \tag{3}$$

which is a linear first-order equation in dependent variable z. This equation can be solved by the method adopted earlier.

The solution to the original problem is given by

$$y = z^{1/(1-n)}. \tag{4}$$

Example

$$3xy' + y + x^2y^4 = 0.$$

Solution:

The given equation can be written in Bernoulli's form as

$$y' + \frac{1}{3x}y = -\frac{x}{3}y^4$$
$$\Rightarrow y^{-4}y' + \frac{1}{3x}y^{-3} = -\frac{x}{3}$$

Putting $z = y^{-3}$:

$$z' = -3y^{-4}y'.$$

Then the ODE becomes

$$z' - \frac{1}{x}z = x.$$

The integrating factor is

$$\text{I.F.} = e^{-\int \frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}.$$

Therefore, the solution can be written as

$$\begin{aligned} z \times \frac{1}{x} &= \int \frac{x}{x}dx + c = x + c \\ \Rightarrow z &= x(c+x). \end{aligned}$$

Therefore, the required solution is

$$y^{-3} = x(c+x),$$

or,

$$y^3 = \frac{1}{x(c+x)}.$$

Riccati equation

Another first-order equation of importance is

$$y' = P(x)y^2 + Q(x)y + R(x), \tag{5}$$

where P, Q and R are functions of x. For $R(x) = 0$, equation (5) reduces to Bernoulli's equation.

For $R(x) \neq 0$:

we develop the following method for finding solution to (5).

Let $y = u(x)$ be a known solution of (5). Then put

$$y = u + \frac{1}{z}, \tag{6}$$

where $z(x)$ is an unknown function.

Then

$$y' = u' - \frac{1}{z^2}z'.$$

Then equation (5) becomes

$$u' - \frac{1}{z^2}z' = P(u^2 + \frac{1}{z^2} + \frac{2u}{z}) + Q(u + \frac{1}{z}) + R.$$

We obtain

$$u' - \frac{1}{z^2}z' = (Pu^2 + Qu + R) + P(\frac{1}{z^2} + \frac{2u}{z}) + \frac{Q}{z}. \tag{7}$$

Since u satisfies $u' = Pu^2 + Qu + R$, we have

$$-\frac{dz}{dx} = P(1 + 2uz) + Qz, \tag{8}$$

which is a linear differential equation in z since u is a known quantity.

(8) can be seen as

$$\frac{dz}{dx} + (2Pu + Q)z = P(x), \tag{9}$$

which can be solved by using the usual procedure and then get $y = u + \frac{1}{z}$ as the solution.

Example:

$$y' = y^2 + (1-2x)y + (x^2 - x + 1); \quad u = x.$$

Solution:

Here $P = 1$, $Q = 1 - 2x$ and $R = x^2 - x + 1$.

Let

$$y = x + \frac{1}{z}$$

so that

$$y' = 1 - \frac{1}{z^2} z'.$$

Therefore

$$1 - \frac{1}{z^2} z' = (x + \frac{1}{z})^2 + (1-2x)(x + \frac{1}{z}) + (x^2 - x + 1)$$
$$\Rightarrow 1 - \frac{1}{z^2} z' = x^2 + \frac{2x}{z} + \frac{1}{z^2} + x + \frac{1}{z} - 2x^2 - \frac{2x}{z} + x^2 - x + 1.$$

This gives us

$$-\frac{1}{z^2}z' = \frac{1}{z} + \frac{1}{z^2}$$
$$\Rightarrow z' = -z - 1.$$

Integrating with integrating factor e^x:

$$z = Ae^{-x} - 1.$$

The solution can be written as

$$y = x + \frac{1}{z}$$
$$\Rightarrow y = x + \frac{1}{Ae^{-x} - 1},$$

where A is a constant.

III. Equation of Clairaut

is as follows:

$$y = xy' + f(y'). \tag{10}$$

The solution of equation (10) can be written as

$$y = mx + f(m), \tag{11}$$

where m is a parameter.

Since $y' = m$, by eliminating m from (10), we get the ODE (10).

Example:

$$y = xy' + (y')^2.$$

Solution:

Let $y' = v$ so that $y = xv + v^2$.

Differentiating

$$y' = v + xv' + 2vv'.$$

Since $y' = v$, we have

$$v'(x + 2v) = 0.$$

There are two cases to consider:

Case I: $v' = 0 \Rightarrow v = m$. This gives the general solution

$$y = mx + m^2.$$

Case II:

$$x + 2v = 0 \quad \Rightarrow v = -\frac{x}{2}.$$

Substituting in $y = xv + v^2$ yields

$$y = -\frac{x^2}{4}.$$

This solution satisfies the differential equation but cannot be obtained from $y = mx + m^2$ by giving particular values of m.

The solution $y = -x^2/4$ is usually called a singular solution which does not contain any arbitrary constant.

Analytically it can be proved that $y = mx + m^2$ is a tangent line which touches the parabola $y = -x^2/4$. Thus, for different values of m, we get different tangent lines, and the parabola $y = -x^2/4$ which envelopes all tangent lines is called the envelope.

The general second-order linear differential equation is

$$y'' + P(x)y' + Q(x)y = R(x), \tag{12}$$

where $P(x)$, $Q(x)$ and $R(x)$ are functions of x alone or may be constants.

The term $R(x)$ in equation (12) is isolated from the others and written on the right side because it does not contain the dependent variable y or any of its derivatives.

If $R(x)$ is identically zero, then (12) reduces to

$$y'' + P(x)y' + Q(x)y = 0 \tag{13}$$

which is called a homogeneous equation. If $R(x)$ is not identically zero, then (12) is said to be non-homogeneous.

If y_c is the solution (called complementary function) corresponding to the homogeneous equation and y_p denotes a particular solution corresponding to the non-homogeneous part, the total solution is given by $y = y_c + y_p$.

The general solution of the homogeneous equation:

Definition: If two functions $f(x)$ and $g(x)$ are defined on an interval $[a, b]$ and have the property that one is a constant multiple of the other, then they are said to be linearly dependent on $[a, b]$. Otherwise, i.e., if neither is a constant multiple of the other, they are called linearly independent.

Theorem:

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{14}$$

on the interval $[a, b]$, then

$$c_1 y_1(x) + c_2 y_2(x) \tag{15}$$

is the general solution of equation (14) on $[a, b]$, in the sense that every solution of (14) on this interval can be obtained from (15) by a suitable choice of the arbitrary constants c_1 and c_2.

Boundary conditions are conditions prescribed on the boundary.

Boundary may be a boundary with respect to any of the independent variables.

Initial conditions are conditions prescribed at one point only.

These conditions are in terms of some form of the dependent variable at some specific value of the independent variable.

The main component of this type of problems is what is called Governing Equation.