MA542: Differential Equations
Lecture - 38

18/04/2022

Finite Vibrating String with no External Force

- Recall the finite string problem in a computational domain $(x, t) \in [0, L] \times [0, \infty)$
  - The governing equation:

$$u_{tt} = c^2 u_{xx}, \;\; (x, t) \in (0, L) \times (0, \infty). \tag{1}$$

  - The boundary conditions for all $t > 0$:

$$u(0, t) = 0, \quad u(L, t) = 0. \tag{2}$$

  - The initial conditions for $0 \leq x \leq L$:

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \tag{3}$$

Formal Solution of the Finite Vibrating String Problem

The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right], \tag{4}$$

with

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}\,dx,\ \ n = 1,2,3,\ldots, \\ B_n &= \frac{2}{n\pi c}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx,\ \ n = 1,2,3,\ldots. \end{aligned}$$

IBVP for Vibrating string (Contd.)

The individual displacement for each n in (4) is referred as
the n-th eigenfunction or the n-th normal mode of the vibrating string.

The n-th normal mode vibrates with a period of $\frac{2L}{nc}$ seconds
which corresponds to a frequency of $\frac{nc}{2L}$ cycles per second.

Since $c^2 = T/\rho$, where T is the tension and ρ is the density of the string,
the frequency is

$$\frac{n}{2L}(T/\rho)^{1/2}.$$

Hence, if a string on a musical instrument is vibrating in a normal mode,
its pitch may be sharpened (frequency increased) by either decreasing the length L of the string or increasing the tension in the string.

IBVP for Vibrating string (Contd.)

The first normal mode $n = 1$ vibrates with
the lowest frequency

$$\frac{1}{2L}(T/\rho)^{1/2}.$$

This is called the *fundamental frequency* of the string.

If the string is made to vibrate in a higher mode,
the frequency is increased by an integer multiple and this corresponds to the production of a musical harmonic or overtone.

When a vibrating system has multiples of fundamental frequency, say in a violin, then music is produced.

When a vibrating system has frequencies which are not integer multiples of fundamental frequency, then noise is produced.

IBVP for Vibrating string (Contd.)

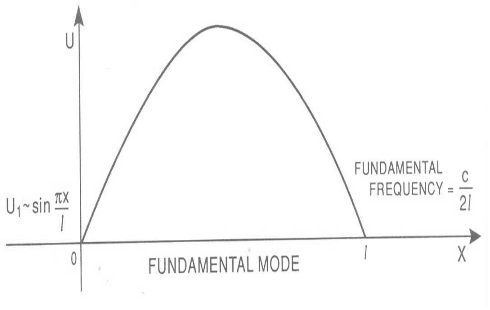

Figure : Fundamental mode of a vibrating string

IBVP for Vibrating string with gravity(Contd.)

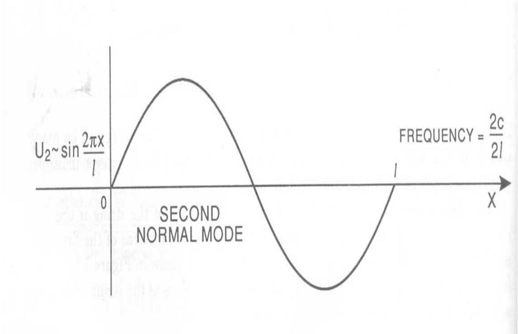

Figure : Second normal mode of a vibrating string

IBVP for Vibrating string (Contd.)

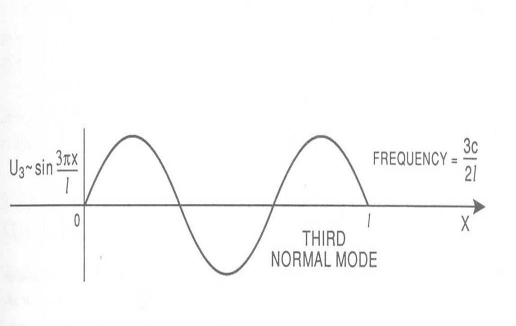

Figure : Third normal mode of a vibrating string

IBVP for Vibrating string (Contd.)

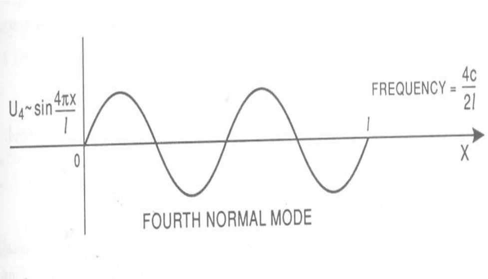

Figure : Fourth normal mode of a vibrating string

IBVP for Vibrating string (Contd.)

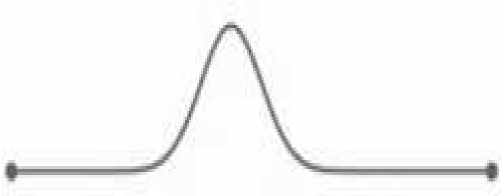

Figure : A pulse traveling through a string with fixed endpoints as modeled by the wave equation

Solution of the Finite Vibrating String Problem: Example

Example: For a string of length L stretched between the points $x = 0$ and $x = L$, find the vibration in the string subject to the following initial conditions:

$$u(x,0) = \sin(\pi x/L) + 1/2\sin(3\pi x/L), \quad u_t(x,0) = 0.$$

Solution: Here, initial conditions are

$$\phi(x) = \sin(\pi x/L) + 1/2\sin(3\pi x/L), \quad \psi(x) = 0.$$

Therefore, $B_n = 0$.
and

$$A_n = \frac{2}{L}\int_0^L \left(\sin\frac{\pi x}{L} + \frac{1}{2}\sin\frac{3\pi x}{L}\right)\sin\frac{n\pi x}{L}dx.$$

Due to the orthogonality of the set $\{\sin\frac{n\pi x}{L} : n = 1,\ 2, \ldots\}$,
only A_1 and A_3 are non-zero, and they are found as $A_1 = 1, A_3 = 1/2$.

Therefore, the solution of the IBVP is

$$u(x,t) = \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + \frac{1}{2}\sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L}.$$

Finite Vibrating String with Gravity

We consider an external force due to the gravitational acceleration g only (consider a string oriented horizontally). Then the one-dimensional wave equation becomes

$$u_{tt} = c^2 u_{xx} - g, \quad 0 < x < L, \quad t > 0. \tag{5}$$

We seek to find

the displacement of the string at any position x and at any time t subject to the following boundary condition (for $t > 0$) and initial conditions ($0 \leq x \leq L$):

$$\begin{aligned} u(0,t) &= 0, && \text{(6a)} \\ u(L,t) &= 0, && \text{(6b)} \end{aligned}$$

and

$$\begin{aligned} u(x,0) &= \phi(x), && \text{(7a)} \\ u_t(x,0) &= \psi(x). && \text{(7b)} \end{aligned}$$

Finite Vibrating String with Gravity (Contd.)

Due to the presence of the term g in equation (5), the equation has now become non-homogeneous and hence the direct application of the method of separation of variables will not work.

Now we intend to convert the given problem into two known solvable problems:

one would resemble the problem with homogeneous equation and the other will take care of the nonhomogeneous term.

Seek a solution in the form:

$$u(x, t) = v(x, t) + h(x), \tag{8}$$

where $h(x)$ is an unknown function of x alone.

Now, using (8) in (5), we obtain

$$v_{tt} = c^2[v_{xx} + h''(x)] - g. \tag{9}$$

Finite Vibrating String with Gravity (Contd.)

We select function h to take care of the non-homogeneous term g such that

$$c^2 h''(x) = g, \tag{10}$$

and then in turn $v(x, t)$ satisfies homogeneous wave equation

$$v_{tt} = c^2 v_{xx}. \tag{11}$$

Both functions v and h are related by boundary conditions

$$v(0, t) + h(0) = 0, \tag{12a}$$
$$v(L, t) + h(L) = 0, \tag{12b}$$

and initial conditions

$$v(x, 0) + h(x) = \phi(x), \tag{13a}$$
$$v_t(x, 0) = \psi(x). \tag{13b}$$

Finite Vibrating String with Gravity (Contd.)

Since, h is a user defined function, we set

$$h(0) = 0 \;\; \& \;\; h(L) = 0. \tag{14}$$

Now the original non-homogeneous problem can be conveniently split into two problems:

Problem I:

$$\begin{aligned} &c^2 h''(x) = g, \\ &h(0) = 0 = h(L). \end{aligned}$$

Problem II:

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \\ v(0,t) &= 0 = v(L,t), \\ v(x,0) &= \phi(x) - h(x), \;\; v_t(x,0) = \psi(x). \end{aligned}$$

Finite Vibrating String with Gravity (Contd.)

The solution for Problem I can be easily found by integrating $h''(x)$ twice:

$$h(x) = \frac{gx^2}{2c^2} + Ax + B.$$

Upon using the conditions $h(0) = 0 = h(L)$, we get

$$B = 0 \ \& \ A = -gL/(2c^2).$$

Hence

$$h(x) = -g\frac{(L-x)x}{2c^2}. \qquad (15)$$

Finite Vibrating String with Gravity (Contd.)

The solution of Problem II is known to us, which is

$$v(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n \cos(\frac{n\pi ct}{L}) + B_n \sin(\frac{n\pi ct}{L})\right], \qquad (16)$$

where A_n and B_n are given, respectively, by

$$A_n = \frac{2}{L}\int_0^L [\phi(x) - h(x)] \sin(\frac{n\pi x}{L})\, dx, \;\; n = 1, 2, 3, \ldots, \qquad (17)$$

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x) \sin(\frac{n\pi x}{L})\, dx, \;\; n = 1, 2, 3, \ldots \qquad (18)$$

Hence the solution $u(x,t)$ for our IBVP is given by the sum of (15) and (16).

Remark: Clearly, the splitting method would be applicable only when non-homogeneous term is a constant or a function of x.

Duhamel's Principle: Finite String Problem

If $v(x,t,s)$ is the solution of the IBVP

$$v_{tt} - c^2 v_{xx} = 0, \quad (x,t) \in (0,L) \times (0,\infty), \tag{19}$$

$$\text{with BCs } v(0,t,s) = 0, \quad v(L,t,s) = 0, \ t > 0, \ s > 0, \ \text{ and} \tag{20}$$

$$\text{with ICs } v(x,0,s) = 0, \quad v_t(x,0,s) = f(x,s), \quad s > 0, \tag{21}$$

then $u(x,t)$ defined by

$$u(x,t) = \int_0^t v(x, t-\tau, \tau) d\tau \tag{22}$$

is the solution to the non-homogeneous problem

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad (x,t) \in (0,L) \times (0,\infty), \tag{23}$$

$$\text{with BCs } u(0,t) = 0, \quad u(L,t) = 0, \ t > 0, \ \text{ and} \tag{24}$$

$$\text{with ICs } u(x,0) = 0, \quad u_t(x,0) = 0. \tag{25}$$

Finite String Problem:Duhamel's Principle

Example: Find $u(x,t)$ such that

$$u_{tt} - u_{xx} = t \sin \frac{\pi x}{L}, \;\; (x,t) \in (0,L) \times (0,\infty), \tag{26}$$

$$\text{with ICs } \; u(x,0) = 0, \;\; u_t(x,0) = 0, \;\; x \in (0,L), \tag{27}$$

$$\text{with BCs } \; u(0,t) = 0, \;\; u(L,t) = 0, \;\; t > 0. \tag{28}$$

Suppose $v(x,t,s)$ is a solution to the user defined problem:

$$v_{tt} - u_{xx} = 0, \;\; (x,t) \in (0,L) \times (0,\infty), \tag{29}$$

$$\text{with ICs } \; v(x,0) = 0, \;\; v_t(x,0) = s \sin \frac{\pi x}{L}, \;\; x \in (0,L), \; s > 0. \tag{30}$$

$$\text{with BCs } \; v(0,t) = 0, \;\; v(L,t) = 0, \;\; t > 0. \tag{31}$$

Finite String Problem: Duhamel's Principle

The solution is given by

$$v(x,t,s) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n \cos\frac{n\pi t}{L} + B_n \sin\frac{n\pi t}{L}\right] \tag{32}$$

with

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}\,dx = 0,\ \ n = 1,2,3,\ldots \\ B_n &= \frac{2}{n\pi}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx,\ \ n = 1,2,3,\ldots \\ &= \frac{2}{n\pi}\int_0^L s\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx. \end{aligned}$$

Finite String Problem: Duhamel's Principle

Thus, $B_1 = \frac{sL}{\pi}$ and $B_n = 0, \; n \neq 1$, and hence

$$v(x,t,s) = \frac{sL}{\pi}\sin\frac{\pi x}{L}\sin\frac{\pi t}{L}.$$

Then solution $u(x,t)$ of the given problem is obtained as

$$\begin{aligned} u(x,t) &= \int_0^t v(x,t-\tau,\tau)d\tau \\ &= \int_0^t \frac{\tau L}{\pi}\sin\frac{\pi x}{L}\sin\frac{\pi(t-\tau)}{L}d\tau \\ &= \frac{L}{\pi}\sin\frac{\pi x}{L}\int_0^t \tau\sin\frac{\pi(t-\tau)}{L}d\tau. \end{aligned}$$

Infinite String Problem: Duhamel's Principle

Infinite String Problem: Application of Duhamel's Principle

Find $u(x, t)$ such that

$$u_{tt} - u_{xx} = x - t, \quad (x, t) \in (-\infty, \infty) \times (0, \infty) \tag{33}$$

$$\text{ICs} \quad u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad x \in (-\infty, \infty), \; s > 0. \tag{34}$$

Infinite String Problem: Duhamel's Principle

Solution: Suppose $v(x, t, s)$ solves following user-defined problem

$$v_{tt} - v_{xx} = 0, \;\; (x, t) \in (-\infty, \infty) \times (0, \infty) \tag{35}$$

$$\text{ICs} \;\; v(x, 0) = 0, \;\; v_t(x, 0) = f(x, s) = x - s, \;\; x \in (-\infty, \infty), \; s > 0. \tag{36}$$

D'Alembert's solution is given by

$$v(x, t, s) = \frac{1}{2}\int_{x-t}^{x+t} f(\tau, s) d\tau = \frac{1}{2}\int_{x-t}^{x+t} (\tau - s) d\tau \tag{37}$$

$$= \frac{1}{2}\left[\frac{\tau^2}{2} - s\tau\right]_{x-t}^{x+t} = xt - ts = t(x - s). \tag{38}$$

Solution to the non-homogeneous problem is given by

$$u(x, t) = \int_0^t v(x, t - \tau, \tau) d\tau \tag{39}$$

$$= \int_0^t 2(t - \tau)(x - \tau) d\tau = -\frac{t^3}{6} + \frac{t^2 x}{2}. \tag{40}$$