MA542 Differential Equations
Lecture - 37

12/04/2022

IBVP for Vibrating string with no external forces

- We consider the problem in a computational domain

$$(x, t) \in [0, L] \times [0, \infty)$$

- The IBVP under consideration consists of the following:
- The governing equation:

$$u_{tt} = c^2 u_{xx}, \quad (x, t) \in (0, L) \times (0, \infty). \tag{1}$$

▶ The boundary conditions for all $t > 0$:

$$u(0, t) = 0, \quad u(L, t) = 0. \tag{2}$$

▶ The initial conditions for $0 \leq x \leq L$ are

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \tag{3}$$

Separation of variables method

- The main idea of this method is to convert the given partial differential equation into several ordinary differential equations. How?
- The solution is assumed to consist of the product of two or more functions.
- The number of functions involved depends on the number of independent variables.
- For wave equation $u = u(x, t)$, so we will assume a solution in the form $u(x, t) = X(x)T(t)$, where X is a function of x only and T is a function of T only.
- Substituting this solution in the given equation we will have a pair of ODEs.
- Note that this method can be used only for bounded domains so that boundary conditions are prescribed.

IBVP for Vibrating string with no external forces (Contd.)

- Recall the wave equation

$$u_{tt} - c^2 u_{xx} = 0. \tag{4}$$

- Assume a solution of the form

$$u(x,t) = X(x)T(t). \tag{5}$$

Here, $X(x)$ is function of x alone and $T(t)$ is a function of t alone.

- Substituting (5) in equation (4)

$$XT'' = c^2 X'' T. \tag{6}$$

IBVP for Vibrating string with no external forces (Contd.)

- Separating the variables

$$\frac{X''}{X} = \frac{T''}{c^2 T}.$$

- Here the left side is a function of x and the right side is a function of t.
- The equality will hold only if both are equal to a constant, say, k.
- We get two differential equations as follows:

$$X'' - kX = 0, \tag{7a}$$
$$T'' - c^2 kT = 0. \tag{7b}$$

IBVP for Vibrating string with no external forces (Contd.)

- Since k is any constant,
  - it can be zero, or
  - it can be positive, or
  - it can be negative.

- Consider all the possibilities and
  examine what value(s) of k lead to a non-trivial solution.

IBVP for Vibrating string with no external forces (Contd.)

Case I: $k = 0$

- In this case the equations (7) reduce to

$$X'' = 0, \quad \text{and} \quad T'' = 0$$

- Giving rise to solutions

$$X(x) = Ax + B, \quad T(t) = Ct + D.$$

- Boundary conditions

$$u(0,t) = u(L,t) = 0,$$

leads to $X(x) = 0$. Hence $u = X(x)T(t) = 0$.

- This case of $k = 0$ is rejected since it gives rise to trivial solution only.

IBVP for Vibrating string with no external forces (Contd.)

Case II: $k > 0$**, let** $k = \lambda^2$ **for some** $\lambda > 0$**.**

- In this case the equations (7) reduce to the equations

$$X'' - \lambda^2 X = 0, \quad \text{and} \quad T'' - c^2\lambda^2 T = 0$$

- Giving rise to solutions

$$\begin{aligned} X(x) &= Ae^{\lambda x} + Be^{-\lambda x}, \\ T(t) &= Ce^{c\lambda t} + De^{-c\lambda t}. \end{aligned}$$

- Therefore

$$u(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{c\lambda t} + De^{-c\lambda t}).$$

IBVP for Vibrating string with no external forces (Contd.)

- Using boundary condition $u(0,t) = 0$,

$$A + B = 0, \quad B = -A.$$

- Using boundary condition $u(L,t) = 0$,

$$(Ae^{\lambda L} + Be^{-\lambda L})(Ce^{c\lambda t} + De^{-c\lambda t}) = 0.$$

- The t part of the solution cannot be zero as it will lead to $T(t) = 0$ and then case $k > 0$ will be rejected straightway.

- Then we must have

$$A(e^{\lambda L} - e^{-\lambda L}) = 0,$$

- Which leads to $A = 0$ as $\lambda \neq 0$.

- $k > 0$ also gives rise to trivial solution:
  so $k > 0$ is also rejected.

IBVP for Vibrating string with no external forces (Contd.)

Case III: $k < 0$, **let** $k = -\lambda^2$ **for some** $\lambda > 0$.

- In this case equations (7) reduce to the equations

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad T'' + c^2\lambda^2 T = 0$$

- Giving rise to solutions

$$\begin{aligned} X(x) &= A\cos\lambda x + B\sin\lambda x, \\ T(t) &= C\cos(c\lambda t) + D\cos(c\lambda t). \end{aligned}$$

- Hence

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos(c\lambda t) + D\sin(c\lambda t)).$$

- Using boundary condition $u(0,t) = 0$, $A = 0$.

IBVP for Vibrating string with no external forces (Contd.)

- Using boundary condition $u(L, t) = 0$,

$$B \sin \lambda L = 0.$$

- $B \neq 0$ as that will lead to a trivial solution.

- Hence we must have

$$\sin \lambda L = 0.$$

- Which gives us

$$\lambda = \frac{n\pi}{L} = \lambda_n, \ n = 1, 2, 3, \ldots.$$

- These λ_n's are called eigenvalues and note that corresponding to each n there will be an eigenvalue.

IBVP for Vibrating string with no external forces (Contd.)

- Accordingly, the solution is

$$
\begin{aligned}
u(x,t) &= (A\cos\lambda x + B\sin\lambda x)(C\cos(c\lambda t) + D\sin(c\lambda t)) \\
&= \sin\lambda_n x(BC\cos(c\lambda_n t) + BD\sin(c\lambda_n t)) \\
&= \sin\frac{n\pi x}{L}\left[A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right], \quad \lambda_n = \frac{n\pi}{L} \\
&= u_n(x,t).
\end{aligned}
$$

- The solution corresponding to each eigenvalue is called an eigenfunction

- Thus, $u_n(x,t)$ is the eigenfunction corresponding to the eigenvalue λ_n.

IBVP for Vibrating string with no external forces (Contd.)

- Since the wave equation is linear and homogeneous, any linear combination will also be a solution

- Hence, we can expect the solution in the following form:

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\
&= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right], \quad (8)
\end{aligned}
$$

- provided

i. A_n and B_n are determined uniquely and
ii. each of the resulting series for those coefficients converges, and
iii. the limit of the series is twice continuously differentiable with respect to x and t so that it satisfies the equation $u_{tt} - c^2u_{xx} = 0$,

IBVP for Vibrating string with no external forces (Contd.)

- What is the idea?

- First, we assume that (!)
  (**i**) the infinite series converges for some A_n and B_n

  (**ii**) term-wise differentiation with respect to t is possible and it converges

  - Next, we calculate A_n and B_n using the given initial conditions.

- Once, both the coefficients A_n and B_n are calculated, we then prove that the series actually holds following properties

  i. the resulting series for those coefficients converge, and

  ii. the limit of the series is twice continuously differentiable with respect to x and t, and it satisfies the partial differential equation.

IBVP for Vibrating string with no external forces (Contd.)

- Using the initial condition $u(x,0) = \phi(x)$

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{9}$$

- This series can be recognized as the half-range sine expansion of a function $\phi(x)$ defined in the range $(0, L)$.

- A_n can be obtained by multiplying equation (9) by $\sin \frac{n\pi x}{L}$ and integrating with respect to x from 0 to L.

- Therefore

$$A_n = \frac{2}{L}\int_0^L \phi(x) \sin \frac{n\pi x}{L}\, dx,\ n = 1, 2, 3, \ldots \tag{10}$$

- Here, we have used the fact that

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} dx = L.$$

IBVP for Vibrating string with no external forces (Contd.)

- To use the other initial condition $u_t(x,0) = \psi(x)$, we need to differentiate (8) w.r.t. t to get

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(\frac{n\pi c}{L}\right)\left[-A_n \sin\frac{n\pi ct}{L} + B_n \cos\frac{n\pi ct}{L}\right].$$

- Then

$$\psi(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L}.$$

- Similarly

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx,\ n = 1, 2, 3, \ldots \tag{11}$$

IBVP for Vibrating string with no external forces (Contd.)

- Now, consider the infinite series

$$\sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n\cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right], \qquad (12)$$

with

-

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}\,dx,\ \ n = 1,2,3,\ldots \qquad (13)$$

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx,\ \ n = 1,2,3,\ldots. \qquad (14)$$

- Clearly, the series satisfies both initial and boundary conditions.
- In fact, it can be proved that it is the solution of the finite vibrating string problem.