MA 542 Differential Equations
Lecture - 36

05/04/2022

IBVP for finite vibrating string with no external forces

- We consider the problem in a computational domain

$$(x, t) \in [0, L] \times [0, \infty)$$

- The IBVP under consideration consists of the following:
- The governing equation:

$$u_{tt} = c^2 u_{xx}, \quad (x, t) \in (0, L) \times (0, \infty). \tag{1}$$

▶ The boundary conditions for all $t > 0$:

$$u(0, t) = 0, \quad u(L, t) = 0. \tag{2}$$

▶ The initial conditions for $0 \leq x \leq L$ are

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \tag{3}$$

HISTORICAL REMARKS on Fourier Series

- The theory of Fourier series had its historical origin in the middle of the eighteenth century, when several mathematicians were studying the vibrations of stretched strings.
- For the case of a string stretched between the points $x = 0$ and $x = \pi$, Daniel Bernoulli (in 1753) gave the solution of (1) as a series of the form

$$u(x,t) = b_1 \sin x \cos ct + b_2 \sin 2x \cos 2ct + \cdots . \tag{4}$$

▶ Observe that:

- A typical term of this series, $b_n \sin nx \cos nct$, is a solution of (1).
- Further, every finite sum of such terms is a solution.
- The series (4) is also a solution if term-by-term differentiation of the series is justified.

HISTORICAL REMARKS on Fourier Series (contd.)

- When $t = 0$, the series (4) reduces to

$$u(x, 0) = b_1 \sin x + b_2 \sin 2x + \dots \tag{5}$$

- This should give the initial shape of the string, that is, the curve

$$u = u(x, 0) = \phi(x).$$

- Thus, we should have

$$\phi(x) = b_1 \sin x + b_2 \sin 2x + \dots \tag{6}$$

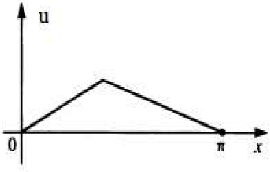
- It is clear on physical grounds that there is a great amount of freedom in the way the string can be constrained in its initial position.

HISTORICAL REMARKS on Fourier Series (contd.)

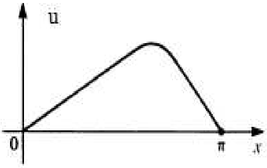
▶ For Example:

(a) If the string is plucked aside at a single point, then the shape will be a broken line,

(b) If it is pushed aside by using a circular object of some kind, then the shape will be partly a straight line, partly an arc of a circle, and partly another straight line.



(a)



(b)

- It is reasonable to ask whether the single analytic expression (6) could represent a straight line on part of the interval $[0, \pi]$, a circle on another part, and a second straight line on still another part?

Fourier Series *on its way*

- Therefore, as a result of mathematically analyzing this physical problem, Bernoulli arrived at an idea that has had very far-reaching influence on the history of mathematics and physical science, namely, the possibility that any function can be expanded in a trigonometric series of the form (6).
- However, D'Alembert (in 1747) and Euler (in 1748) rejected Bernoulli's idea, and for essentially the same reason.
- The controversy bubbled on for many years, and in the absence of mathematical proofs, no one could convince anyone else to his way of thinking.

Fourier Series *on its way*

- In 1807, the French physicist-mathematician Jean Baptiste Fourier announced in this connection that an arbitrary function f(x) can be represented in the form (6).
- He supplied no proofs, but instead heaped up the evidence of many solved problems and many convincing specific expansions - so many, indeed, that the mathematicians of that time began to spend more effort on proving, rather than disproving, his conjecture.
- The first major result of this shift in the winds of opinion was the classical paper of Dirichlet in 1829, in which he proved with full mathematical rigour that such a series exists.

Let us know Fourier

Jean Baptiste Joseph Fourier (March 21, 1768 – May 16, 1830) was a French mathematician and physicist who is best known for initiating the investigation of Fourier series and their application to problems of heat flow. He is also known for Fourier transform and Fourier's heat law.

Some Facts:

- Alma mater: Ecole Normale Suprieure
- Academic advisor: Joseph-Louis Lagrange
- Notable students:
  Peter Gustav Lejeune Dirichlet
  Claude-Louis Navier
  Giovanni Plana

Let us know Fourier

- Profound study of nature is the most fertile source of mathematical discoveries.
- The differential equations of the propagation of heat express the most general conditions, and reduce the physical questions to problems of pure analysis, and this is the proper object of theory.
- Heat, like gravity, penetrates every substance of the universe, its rays occupy all parts of space. The object of our work is to set forth the mathematical laws which this element obeys. The theory of heat will hereafter form one of the most important branches of general physics.
- Mathematical analysis is as extensive as nature herself.

Fourier Series: *Orthogonal Sets*

- We begin our treatment with some observations: For $m, n = 1, 2, 3, \ldots$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}dx = 0 \;\;\&\;\; \int_{-L}^{L} \sin\frac{n\pi x}{L}dx = 0, \tag{7}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 0, \tag{8}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{9}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{10}$$

- Orthogonal Functions: A set of functions $\{f_n(x)\}_{n=1}^{\infty}$ is said to be an orthogonal set on the interval $[a, b]$ if

$$\int_a^b f_n(x)f_m(x)dx = 0, \quad m \neq n. \tag{11}$$

Fourier Series: *Orthogonal Sets*

- For an orthogonal set $\{f_n(x)\}_{n=1}^{\infty}$ in $[a, b]$ and a given function $f(x)$, if we have

$$
\begin{aligned}
f(x) &= c_1 f_1(x) + c_2 f_2(x) + c_3 f(x) + \ldots, \quad \text{then} \\
c_n &= \frac{\int_a^b f(x) f_n(x) dx}{\int_a^b f_n(x)^2 dx}, \quad n = 1, 2, \ldots.
\end{aligned}
$$

  - We have already seen that the set of trigonometric functions

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$

    is orthogonal on $[-\pi, \pi]$.

- Thus, if we have

$$f(x) = c_0 + \sum_{n=1}^{\infty} \Big\{ a_n \cos nx + b_n \sin nx \Big\},$$

then all coefficients c_0, a_n and b_n can be determined uniquely.

Fourier Series in $[-\pi, \pi]$

- An infinite series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big\{ a_n \cos nx + b_n \sin nx \Big\}, \qquad (12)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n = 0, 1, 2, \ldots, \qquad (13)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n = 1, 2, \ldots, \qquad (14)$$

is called a Fourier Series.

- Terms a_n and b_n are called Fourier coefficients.

Fourier Series in $[-L, L]$

- The infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right], \tag{15}$$

▶with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0,\ 1,\ 2, \ldots \quad \text{and} \tag{16}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1,\ 2, \ldots \tag{17}$$

is called the Fourier series (FS) of $f(x)$.

- Terms a_n and b_n are called Fourier coefficients.

Remark

The Fourier series of a function is defined whenever the integrals in (16) and (17) have meaning. This is certainly the case if $f(x)$ is continuous on the interval (open or closed).

Fourier Series

- However, the integrals also have meaning when $f(x)$ has jump discontinuities as in the following function:

$$f(x) = \begin{cases} f_1(x), & -L < x < x_1, \\ f_2(x), & x_1 < x < x_2, \\ f_3(x), & x_2 < x < L. \end{cases}$$

- Then the Fourier coefficients a_n and b_n will be given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{x_1} f_1(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{x_1}^{x_2} f_2(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{x_2}^{L} f_3(x)\cos\frac{n\pi x}{L}\,dx, \\ b_n &= \frac{1}{L}\int_{-L}^{x_1} f_1(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{x_1}^{x_2} f_2(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{x_2}^{L} f_3(x)\sin\frac{n\pi x}{L}\,dx. \end{aligned}$$

Example

Find the Fourier Series of the function $f(x) = x$ for $-L \leq x \leq L$.

Example (Contd.)

Solution: We first compute the Fourier coefficients a_n for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{L}dx = \frac{x}{n\pi}\sin\frac{n\pi x}{L}\Bigg|_{-L}^{L} - \frac{1}{n\pi}\int_{-L}^{L}\sin\frac{n\pi x}{L}dx \\ &= 0 + \frac{L}{(n\pi)^2}\cos\frac{n\pi x}{L}\Bigg|_{-L}^{L} = 0,\ n = 1, 2, 3, \ldots. \end{aligned}$$

For $n = 0$, verify that $a_n = 0$. Thus, $a_n = 0$, for $n = 0, 1, 2, 3, \ldots$. Next, to compute $b_n,\ n \geq 1$, we have

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} x\sin\frac{n\pi x}{L}dx = \frac{-x}{n\pi}\cos\frac{n\pi x}{L}\Bigg|_{-L}^{L} + \frac{1}{n\pi}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx \\ &= \frac{-2L}{n\pi}\cos(n\pi) + \frac{L}{(n\pi)^2}\sin\frac{n\pi x}{L}\Bigg|_{-L}^{L} = \frac{2L}{n\pi}(-1)^{n+1}. \end{aligned}$$

Example (Contd.)

Thus, the Fourier series of $f(x)$ is given by

$$f(x) \approx \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2L}{n\pi} \sin \frac{n\pi x}{L} = \frac{2L}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin \frac{n\pi x}{L}.$$

Remark

Here, the symbol $\approx$ means that Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right],$$

with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0,\ 1,\ 2, \ldots \quad \textit{and}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1,\ 2, \ldots$$

is an approximation to $f(x)$.

Example

Find the Fourier Series of the function

$$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi \end{cases}$$

Solution: Here $L = \pi$. Note that $f(x)$ is an odd function. Since the product of an odd function and an even function is odd, $f(x)\cos(nx)$ is also an odd function. Hence

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0, \quad n = 0, 1, 2, \ldots.$$

Since $f(x)\sin x$ is an even function (as the product of two odd functions), we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} \sin nx dx$$

Example (Contd.)

Finally, we arrive at

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{2}{\pi}\Big[\frac{-\cos nx}{n}\Big]\Big|_0^{\pi} \\ &= \frac{2}{\pi}\Big[\frac{1}{n} - \frac{(-1)^n}{n}\Big] \\ &= \begin{matrix} 0, & n \text{ even}, \\ \frac{4}{n\pi} & n \text{ odd}. \end{matrix} \end{aligned}$$

Thus, the Fourier series of $f(x)$ is given by

$$f(x) \approx \frac{4}{\pi}\Big[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\Big].$$

Remark

- *If f is any odd function, then its Fourier series consists only of sine terms.*
- *If f is an even function, then its Fourier series consists only of cosine terms.*

Cosine and sine Fourier Expansion

▶Suppose we are given a function f, defined and integrable on the interval $(0, L)$.

▶Define an even extension of f in $(-L, L)$ by

$$\begin{aligned} f_e(x) &= f(x) \text{ if } 0 < x < L, \\ f_e(x) &= f(-x) \text{ if } -L < x < 0. \end{aligned}$$

▶Define the odd extension of f by

$$\begin{aligned} f_O(x) &= f(x) \text{ if } 0 < x < L, \\ f_O(x) &= -f(-x) \text{ if } -L < x < 0. \end{aligned}$$

Remark

- *Then, we define Fourier series of f in $(0, L)$ as the Fourier series of f_e or f_O in $(0, L)$. Fourier expansion in $(0, L)$ is called* **Fourier half range expansion.**
- *Therefore, Fourier half range expansion is either a Fourier cosine or Fourier sine expansion.*

Half-range Series

Example

Find the Fourier cosine series expansion of $f(x) = x,\ 0 < x < 1$.

Solution Hint:

- Construct an even periodic extension of f.
- Suppose, f_e is the even extension of f in $(-1, 1)$.
- Then, evaluate the Fourier cosine expansion of f_e in $(-1, 1)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x.$$

- By definition, the Fourier cosine expansion of f in $(0, 1)$ is the Fourier cosine expansion f_e in $(0, 1)$

Convergence Problem

- Till now we have been using the notation

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right].$$

  - with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0,\ 1,\ 2, \ldots \quad \text{and}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1,\ 2, \ldots$$

- Can we have equality? That is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right].$$

- This leads to the **convergence problem of Fourier series**.