MA542 Differential Equations

Lecture 34

01/04/2022

Method of Characteristics

Recall

Our Equation

$$Au_{xx} + Bu_{xt} + Cu_{tt} + f(x, t, u, u_x, u_t) = 0. \tag{1}$$

After using the transformation

$$\begin{aligned} Au_{xx} + Bu_{xt} + Cu_{tt} &= (Aa^2 + Bab + Cb^2)U_{\xi\xi} \\ &\quad +(2acA + B(ad + bc) + 2Cbd)U_{\xi\eta} \\ &\quad +(Ac^2 + Bcd + Cd^2)U_{\eta\eta}. \end{aligned} \tag{2}$$

Further with $a = 1 = c$, and $C \neq 0$,

$$b = \frac{-B + \sqrt{\mathbb{D}}}{2C}, \quad d = \frac{-B - \sqrt{\mathbb{D}}}{2C} \tag{3}$$

Method of Characteristics

Parabolic Case, $\mathbb{D} = 0$

In this case, equations (3) give

$$b = d = \frac{-B}{2C}.$$

Hence, the resulting transformation

$$\xi = x + bt, \ \eta = x + bt$$

is not invertible.

Observe that if we choose

$$a = c = 1, d = -B/(2C), \quad \text{and} \ \ b = 0,$$

then the coefficients of

$U_{\eta\eta}$ and $U_{\xi\eta}$ in (2) vanish.

Method of Characteristics (Contd.)

Hence

the transformation

$$\xi = x,\ \eta = x - \frac{B}{2C}t$$

transforms equation (1) into

$$U_{\xi\xi} + H(\xi, \eta, U, U_\xi, U_\eta) = 0 \tag{4}$$

where only one double derivative appears.

Check the case when $C = 0$.

Method of Characteristics (Contd.)

Elliptic Case, $\mathbb{D} < 0$:

Now b and d in (3) are complex conjugate numbers.

That is

$$d = \overline{b}.$$

Selecting $a = c = 1$, we obtain a complex transformation:

$$\xi = x + bt,\ \eta = x + \overline{b}t$$

A real transformation can be found by considering real variables α and β:

$$\alpha = \frac{1}{2}(\xi + \eta),\ \ \beta = \frac{1}{2\mathrm{i}}(\xi - \eta).$$

Method of Characteristics (Contd.)

This transforms (1) into

$$U_{\alpha\alpha} + U_{\beta\beta} + K(\alpha, \beta, U, U_\alpha, U_\beta) = 0. \tag{5}$$

where two second partial derivatives (in the new variables) are still present and only the mixed derivative is absent.

Equation (5) is the canonical form of elliptic equations.

We recognize the combination of the second partial derivatives as

the Laplacian operator.

We notice that

even after using transformation, we are still left with two derivatives and this reduction does not help us in finding the solution by simple integration.

Hence this method is not all suitable for elliptic equations.

Method of Characteristics (Contd.)

Generally characteristic coordinates are useful only for hyperbolic equations.

They do not play any important role in elliptic and parabolic equations.

Note that

first order equations, like a reaction-advection equation, are classified as hyperbolic because

they propagate signals like wave equations.

Method of Characteristics (Contd.)

Example:

Find the characteristics of the following equation and reduce it to the appropriate standard form and then obtain the general solution:

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0.$$

Solution:

Here A=3,B=10,C=3 and we have $B^2 - 4CA = 100 - 36 = 64 > 0.$

Hence the equation is of hyperbolic type.

The characteristics are given by

$$\xi = x + \left(\frac{-B + \sqrt{\mathbb{D}}}{2C}\right) y = x - (1/3)y, \;\; \eta = x + \left(\frac{-B - \sqrt{\mathbb{D}}}{2C}\right) y = x - 3y.$$

Here we have $\xi_x = 1, \eta_x = 1, \xi_y = -1/3, \eta_y = -3.$

Method of Characteristics (Contd.)

Therefore, under this transformation:

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta, \\ u_y &= u_\xi \xi_y + u_\eta \eta_y = (1/3)u_\xi - 3u_\eta. \end{aligned}$$

Expressions for the second order derivatives are

$$\begin{aligned} u_{xx} &= u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\ u_{xy} &= u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y \\ &= -(1/3)u_{\xi\xi} - (10/3)u_{\xi\eta} - 3u_{\eta\eta}, \\ u_{yy} &= (1/3)[u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y] - 3[u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y] \\ &= (1/9)u_{\xi\xi} + 2u_{\xi\eta} + 9u_{\eta\eta}. \end{aligned}$$

Method of Characteristics (Contd.)

Substituting these values in the given equation, we obtain

$$u_{\xi\eta} = 0.$$

Now integrating with respect to ξ partially

$$u_\eta = F'(\eta).$$

Again integrating with respect to η partially

$$u = F(\eta) + G(\xi).$$

Hence the solution of the original equation is

$$u(x, y) = F(x - 3y) + G(3x - y),$$

where F and G are arbitrary functions.

Method of Characteristics (Contd.)

Example:

Find the characteristics of the following equation and reduce it to the appropriate standard form and then obtain the general solution:

$$u_{xx} + 4u_{xy} = 0.$$

Solution:

Here A=1,B=4,C=0 and we have $B^2 - 4CA = 16 > 0$.

Hence the equation is of hyperbolic type.

The characteristics are given by

$$\xi = \frac{-B}{A}x + y = -4x + y, \;\; \eta = y.$$

Here we have $\xi_x = -4, \eta_x = 0, \xi_y = 1, \eta_y = 1$.

Method of Characteristics (Contd.)

Therefore, under this transformation:

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x = -4u_\xi, \\ u_y &= u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta. \end{aligned}$$

Expressions for the second order derivatives are

$$\begin{aligned} u_{xx} &= u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x \\ &= 16u_{\xi\xi}, \\ u_{xy} &= u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y \\ &= -4(u_{\xi\xi} + u_{\xi\eta}). \end{aligned}$$

Method of Characteristics (Contd.)

Substituting these values in the given equation, we obtain

$$u_{\xi\eta} = 0.$$

Now integrating with respect to ξ partially

$$u_\eta = F'(\eta).$$

Again integrating with respect to η partially

$$u = F(\eta) + G(\xi).$$

Hence the solution of the original equation is

$$u(x, y) = F(y) + G(y - 4x),$$

where F and G are arbitrary functions.

Method of Characteristics (Contd.)

Example:

Find the characteristics of the following equation and reduce it to the appropriate standard form and then obtain the general solution:

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0.$$

Solution:

$B^2 - 4CA = 0$

Hence the equation is parabolic.

The characteristics are given by

$$\xi = x,\ \eta = x - \frac{B}{2C}y = x - (1/2)y$$

We have

$$\xi_x = 1, \eta_x = 1, \xi_y = 0, \eta_y = -(1/2)$$

Method of Characteristics (Contd.)

Under this transformation:

$$
\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta, \\
u_y &= u_\xi \xi_y + u_\eta \eta_y = -(1/2) u_\eta.
\end{aligned}
$$

The expressions for the second order derivatives:

$$
\begin{aligned}
u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\
u_{xy} &= -(1/2) u_{\xi\eta} - (1/2) u_{\eta\eta}, \\
u_{yy} &= (1/4) u_{\eta\eta}.
\end{aligned}
$$

Method of Characteristics (Contd.)

Substituting these values in the given equation:

$$u_{\xi\xi} = 0.$$

Integrating with respect to ξ partially:

$$u_{\xi} = f(\eta).$$

Again integrating with respect to ξ partially:

$$u = \xi f(\eta) + g(\eta).$$

Hence the solution of the original equation is

$$u(x, y) = x f(x - (1/2)y) + g(x - (1/2)y),$$

where f and g are arbitrary functions.

Method of Characteristics (Contd.)

Example: (Non-homogeneous equation)

Find the characteristics of the following equation and reduce it to the appropriate standard form and then obtain the general solution:

$$u_{xx} - 4u_{xy} + 4u_{yy} = \cos(2x + y).$$

Solution:

The given equation is a parabolic one.

The characteristics are:

$$\xi = x,\ \eta = x - \frac{B}{2C}y = x + (1/2)y$$

The canonical form will be

$$u_{\xi\xi} = \cos\eta.$$

.

Method of Characteristics (Contd.)

Integrating partially with respect to ξ:

$$u_\xi = \xi \cos\eta + f(\eta).$$

Again integrating w.r.t. ξ

$$u(\xi, \eta) = \frac{\xi^2}{2} \cos\eta + \xi f(\eta) + g(\eta).$$

The solution:

$$u(x, y) = x f(y + 2x) + g(y + 2x) + \frac{x^2}{2} \cos(y + 2x),$$

where f and g are arbitrary functions.