MA542 Differential Equations

Lecture 33

28/03/2022

Classification of Second-order PDEs

General second-order linear partial differential equation in two independent variables x, t is of the form

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = G(x,t),$$

where A, B, C, D, E, F are assumed to be constants for the time being, but they can actually be functions of x and t.

The above equation may conveniently be written as

$$Au_{xx} + Bu_{xt} + Cu_{tt} + f(x,t,u,u_x,u_t) = 0, \tag{1}$$

Note

- The expression $Lu \equiv Au_{xx} + Bu_{xt} + Cu_{tt}$ is called the Principal part of the equation.
- Classification of such PDEs are based on this principal part.
- If the problem involves only spatial variables, replace t by y, say.

Classification of Second order PDEs (Contd.)

The classification is based on

the sign of the quantity $\mathbb{D} \equiv B^2 - 4AC$, called the discriminant.

We say

- Equation (1) is called hyperbolic if $\mathbb{D} > 0$.
- Equation (1) is called parabolic if $\mathbb{D} = 0$.
- Equation (1) is called elliptic if $\mathbb{D} < 0$.

Under this classification

- Wave equation $u_{tt} = c^2 u_{xx}$ is hyperbolic.
- Diffusion or heat equation $u_t = \alpha u_{xx}$ is parabolic.
- Laplace's equation $u_{xx} + u_{yy} = 0$ is elliptic.

Classification of Second order PDEs (Contd.)

The idea behind this terminology:

Familiar classification of plane curves.

For example

$Ax^2 + Ct^2 = 1$, where $A, C > 0, B = 0$. Hence $\mathbb{D} < 0$. It gives us a graph which is an ellipse in the xt-plane.

Similarly

$Ax^2 - Ct^2 = 1$, where $\mathbb{D} > 0$. It gives us a graph which is a hyperbola.

In the same manner

$x^2 - 4at = 0$, where $\mathbb{D} = 0$. It gives us a graph which is a parabola.

Classification of Second order PDEs (Contd.)

If A, B, C are not constants, but are functions of x and t,

the discriminant depends on x and t

Same classifications apply to these cases, but

the sign of $\mathbb{D}$ can change, depending upon the domain.

That is

the classification may change when values of the variables change.

For example, consider

$$(x+2)u_{xx} + 2xu_{xt} + tu_{tt} = 0.$$

It is parabolic if $x^2 = (x+2)t$, hyperbolic if $x^2 > (x+2)t$ and elliptic if $x^2 < (x+2)t$.

Classification of Second order PDEs (Contd.)

The principal part Lu in the equation can be simplified for each of the three types

by introducing a new set of variables

This idea of introducing new variables, common in many differential equations,

renders the given differential equation to a much simplified form: enabling integration without much difficulty.

Method of Characteristics

Assume A, B, C to be constants and seek a linear transformation that simplifies Lu:

$$\xi = ax + bt, \quad \eta = cx + dt$$

ξ and η are new independent variables and a, b, c, d are to be determined for different cases.

Assume that

$$ad - bc \neq 0$$

so that the transformation is invertible, that is,

x and t can be solved in terms of ξ and η.

Method of Characteristics (Contd.)

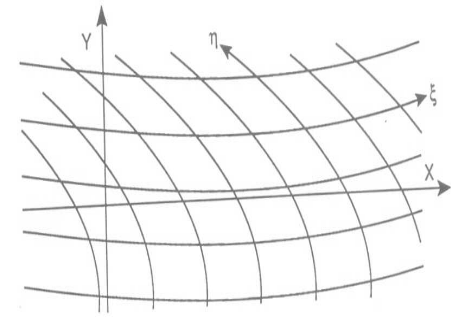

Figure : Characteristic Plane (ξ, η)

Method of Characteristics (Contd.)

The dependent variable u in the new variables will be denoted by

$$U = U(\xi, \eta)$$

that is

$$u(x,t) = U(ax + bt, cx + dt)$$

By chain rule

$$\begin{aligned} u_x &= U_\xi \xi_x + U_\eta \eta_x = aU_\xi + cU_\eta, \\ u_t &= U_\xi \xi_t + U_\eta \eta_t = bU_\xi + dU_\eta. \end{aligned}$$

Method of Characteristics (Contd.)

The second-order partial derivatives can be obtained by another application of the chain rule.

They can be found as

$$\begin{aligned} u_{xx} &= a^2U_{\xi\xi} + 2acU_{\xi\eta} + c^2U_{\eta\eta}, \\ u_{tt} &= b^2U_{\xi\xi} + 2bdU_{\xi\eta} + d^2U_{\eta\eta}, \\ u_{xt} &= abU_{\xi\xi} + (ad+bc)U_{\xi\eta} + cdU_{\eta\eta} \end{aligned}$$

Substituting these into the principal part

$$\begin{aligned} Au_{xx} + Bu_{xt} + Cu_{tt} &= (Aa^2 + Bab + Cb^2)U_{\xi\xi} \\ &\quad +(2acA + B(ad+bc) + 2Cbd)U_{\xi\eta} \\ &\quad +(Ac^2 + Bcd + Cd^2)U_{\eta\eta}. \end{aligned} \tag{2}$$

Method of Characteristics (Contd.)

We can select a, b, c, d so that

some of the second partial derivatives in the new variables disappear.

This process must be handled differently

depending on the sign of the discriminant $\mathbb{D}$.

KEEP IN MIND THAT WHATEVER TRANSFORMATION IS USED, THE TYPE OF THE PDE DOES NOT GET CHANGED

The transformation only allows us to handle a simplified PDE (in new variables)

Method of Characteristics (Contd.)

Hyperbolic Case: $\mathbb{D} > 0$

Let us choose $a = c = 1$

In that case

the coefficients of $U_{\xi\xi}$ and $U_{\eta\eta}$, which have the same form, become quadratic expressions in b and d, respectively. That is $A + Bb + Cb^2 = 0$ and $A + Bd + Cd^2 = 0$.

We can force these coefficients to vanish by choosing (for $C \neq 0$)

$$b = \frac{-B + \sqrt{\mathbb{D}}}{2C}, \quad d = \frac{-B - \sqrt{\mathbb{D}}}{2C}. \tag{3}$$

The remaining coefficient, that of $U_{\xi\eta}$, is nonzero.

Method of Characteristics (Contd.)

Consequently we find that

the transformation

$$\xi = x + \left(\frac{-B+\sqrt{\mathbb{D}}}{2C}\right)t, \quad \eta = x + \left(\frac{-B-\sqrt{\mathbb{D}}}{2C}\right)t \tag{4}$$

transforms the PDE (1) into a simpler equation of the form

$$U_{\xi\eta} + G(\xi, \eta, U, U_\xi, U_\eta) = 0 \tag{5}$$

where only the mixed derivative appears.

Thus this is a significant simplification over (1) in which all the second derivatives occur.

Method of Characteristics (Contd.)

The new transformed equation is called

the canonical form of a hyperbolic equation.

The coordinates ξ and η defined by (4) are called the

characteristic coordinates and $\xi(x,t) = c_1$ and $\eta(x,t) = c_2$ are called characteristic curves.

Note that if $C = 0$, (3) is not valid.

When $C = 0$, choose $b = d = 1$.

Then we have

$$Au_{xx} + Bu_{xt} = (Aa^2 + Ba)U_{\xi\xi} + (2acA + B(a+c))U_{\xi\eta} + (Ac^2 + Bc)U_{\eta\eta}$$

Method of Characteristics (Contd.)

In order that the coefficients of $U_{\xi\xi}$ and $U_{\eta\eta}$ vanish:

$$a = 0, \text{ or } a = \frac{-B}{A}; \ c = 0, \text{ or } c = \frac{-B}{A}$$

Now observe that if we take $a = 0$ and $c = 0$ at the same time, we get

the same characteristics

$$\xi = t, \ \eta = t$$

Similarly if we take $a = \frac{-B}{A}$ and $c = \frac{-B}{A}$ at the same time, we get

the same characteristics again:

$$\xi = \frac{-B}{A}x + t, \ \eta = \frac{-B}{A}x + t$$

Method of Characteristics (Contd.)

But we know that ξ and η cannot have the same expressions.

Hence we consider only those values of a and c at the same time which give

two different expressions for ξ and η.

Any one of the following pairs can be the characteristics:

$$\xi = -\frac{B}{A}x + t, \;\; \eta = t$$
$$\xi = t, \;\; \eta = -\frac{B}{A}x + t$$

Either of them will reduce the given equation again to the simplified one:

$$U_{\xi\eta} + G(\xi, \eta, U, U_\xi, U_\eta) = 0.$$