# MA 542: Differential Equations
# Lecture - 31

24/03/2022

# Integral surfaces through a given curve

- Suppose we have found two solutions

$$\phi(x, y, u) = c_1, \ \ \psi(x, y, u) = c_2 \tag{1}$$

of the auxiliary equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

- Any solution of the corresponding quasi-linear equation $a(x, y, u)\dfrac{\partial u}{\partial x} + b(x, y, u)\dfrac{\partial u}{\partial y} = c(x, y, u)$ is of the form

$$G(\phi, \psi) = 0 \tag{2}$$

- Now, we want to find the integral surface which passes through a given curve $C$ whose parametric equations are

$$x(0) = x_0(s), \ y(0) = y_0(s), \ u(0) = u_0(s),$$

where $s$ is a parameter.

- Recall that, for any point $(x, y, u)$ on the integral surface, we have

$$\phi(x, y, u) = c_1, \;\; \psi(x, y, u) = c_2 \tag{3}$$

- In particular, we must have

$$\phi\{x_0(s), y_0(s), u_0(s)\} = c_1, \; \psi\{x_0(s), y_0(s), u_0(s)\} = c_2. \tag{4}$$

- We eliminate the single variable $s$ to obtain a relation of the type

$$f(c_1, c_2) = 0 = f(\phi, \psi) = f(x, y, u). \tag{5}$$

- **Example:** Find an integral surface of the quasi-linear PDE

$$x(y^2 + u)p - y(x^2 + u)q = (x^2 - y^2)u$$

which contains the straight line $x + y = 0, \; u = 1$.

- **Solution:** The auxiliary equations are

$$\frac{dx}{x(y^2 + u)} = \frac{dy}{-y(x^2 + u)} = \frac{du}{(x^2 - y^2)u}.$$

- Taking

$$\frac{y\ dx + x\ dy}{xy^3 + xyu - yx^3 - yxu} = \frac{du}{(x^2 - y^2)u}$$

will ultimately give rise to

$$\frac{d(xy)}{xy} = -\frac{du}{u},$$

- The solution is

$$xyu = c_1. \tag{6}$$

- Similarly taking

$$\frac{x\ dx + y\ dy}{x^2y^2 + x^2u - y^2x^2 - y^2u} = \frac{du}{(x^2 - y^2)u},$$

$$x^2 + y^2 - 2u = c_2. \tag{7}$$

- For the initial curve $x + y = 0,\ u = 1$, we have the parametric equations

$$x_0(s) = s, y_0(s) = -s, u_0(s) = 1.$$

- Then we have, from (6) and (6), respectively,

$$x_0(s)y_0(s)u_0(s) = c_1 \ \ \& \ \ x_0(s)^2 + y_0(s)^2 - 2u_0(s) = c_2,$$

so that

$$-s^2 = c_1, \ \ \& \ \ 2s^2 - 2 = c_2,$$

and eliminate $s$ from them to have

$$2c_1 + c_2 + 2 = 0.$$

- The desired integral surface is

$$x^2 + y^2 + 2xyu - 2u + 2 = 0.$$

## Example

$$\textbf{PDE}: \ (y+u)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x - y; \ \ \textbf{IC}: \ u(x,1) = 1 + x.$$

**Solution.**

- **Step 1.** (Write the parametric form of the initial curve)
  To solve the IVP, we parameterize the initial curve $\Gamma$ as

$$x_0(s) = s, \ \ y_0(s) = 1, \ \ u_0(s) = 1 + s.$$

- **Step 2.**(Write the initial conditions)

$$x(0) = x_0(s) = s, \ \ y(0) = y_0(s) = 1, \ \ u(0) = u_0(s) = 1 + s.$$

- **Step 3.**(Solve the characteristic equations)

$$\frac{dx}{dt} = y + u, \quad \frac{dy}{dt} = y, \quad \frac{du}{dt} = x - y$$

to have

$$y(t) = y(0)e^t = e^t, \;\; u(t) + x(t) = (u(0) + x(0))e^t = (1 + 2s)e^t.$$

From Step 3, we again obtain

$$\frac{dx}{dt} + x = (2 + 2s)e^t \Rightarrow x(t) = (1 + s)(e^t - e^{-t}) + se^{-t},$$

and

$$u(t) = se^t + e^{-t}.$$

**Step 4.**(If possible get explicit or implicit form of the solution)

Observe that the Jacobian

$$J = \begin{vmatrix} 2+s & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \text{ on } \Gamma.$$

Therefore, transformation $(t, s) \to (x, y)$ is possible around $\Gamma$.

Any point $(x, y, u)$ on the characteristic curve is given by

$$\begin{aligned} x &= x(t) = e^t - e^{-t} + se^t \\ y &= y(t) = e^t, \\ u &= u(t) = se^t + e^{-t}. \end{aligned}$$

Noting that

$$u = u(t) = se^t + e^{-t} = x - e^t + e^{-t} + e^{-t} = x - y + \frac{2}{y}.$$

Note that the solution is not global (it becomes singular on the $x$-axis), but it is well defined near the initial curve.

## Example

$$\textbf{PDE}:\ u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0;\ \ \textbf{IC}:\ u(x,0) = f(x).$$

- **Step 1.** (Write the parametric form of the initial curve)
  To solve the IVP, we parameterize the initial curve as

$$x_0(s) = s, \quad y_0(s) = 0, \quad u_0(s) = f(s).$$

- **Step 2.**(Write the Initial Conditions)

$$x(0) = x_0(s) = s, \ \ y(0) = y_0(s) = 0, \ \ u(0) = u_0(s) = f(s).$$

- **Step 3.**(Solve the Characteristic Equations)

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = 1, \quad \frac{du}{dt} = 0$$

  to have

$$u(t) = u(0) = f(s), \ \ y(t) = t + y(0) = t, \ \ x(t) = f(s)t + x(0).$$

- **Step 4.**(If possible get explicit or implicit form of the solution)
  Note that $(x, y, u)$ on the integral surface satisfies

$$u = u(t) = f(s), \;\; y = y(t) = t, \;\; x = x(t) = f(s)t + s.$$

For the transformation $(t, s) \to (x, y)$, check the transversality condition. Here, $J = -f'(\tau) \neq 0$, along the entire initial curve. So, we can solve for $s$ and $t$ in terms of $x$ and $y$

$$t = y, \;\; s = x - f(s)t = x - uy.$$

- Thus, the solution can also be given in implicit form as

$$u = f(s) = f(x - yu).$$

# Orthogonal Surfaces

- Suppose a surface is characterized by the equation

$$f(x, y, u) = 0. \tag{8}$$

We want to find a surface which cuts the given surface at right angles.

- Assume that the surface with equation

$$u(x, y) - u = 0 = F(x, y, u) \tag{9}$$

cuts each surface of the given system orthogonally.

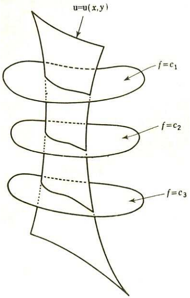


Figure : Orthogonal Surfaces

- Then at point of intersection $(x, y, u)$, we have following quasi-linear PDE
$$\nabla f \cdot \nabla F = f_x u_x + f_y u_y - f_u = 0. \tag{10}$$
- We know that if we want to solve a PDE of the form
$$a u_x + b u_y = c,$$
we solve the equation with the help of the following auxiliary equations:
$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$
- Therefore it is clear that in order to solve (10), we need to solve the following auxiliary equations:
$$\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{du}{f_u}.$$

## Example

Find a surface which intersects the surface $u(x+y) = (3u+1)$ orthogonally and which passes through the circle $x^2+y^2=1,\ u=1$.

- **Solution:** Here $f = u(x+y) - (3u+1)$ and hence we have

$$\frac{\partial f}{\partial x} = u,\ \frac{\partial f}{\partial y} = u, \frac{\partial f}{\partial u} = x+y-3.$$

- The integral curves are given by

$$\frac{dx}{u} = \frac{dy}{u} = \frac{du}{x+y-3}$$

$$\Rightarrow \frac{dx}{u} = \frac{dy}{u} = \frac{du}{x+y-3} = \frac{dx+dy}{2u}.$$

- Taking the first two, we get

$$\phi = x - y = c_1 \tag{11}$$

and taking 3rd and 4th relation

$$\psi = (x+y)^2 - 6(x+y) - u^2 = c_2. \tag{12}$$

- We write the given curve in parametric form as

$$\{(x_0(s), y_0(s), u_0(s)) : s \in J \;\&\; u_0(s) = 1,\ x_0(s)^2 + y_0(s)^2 = 1.\}$$

Observe that

$$\begin{aligned} &2x_0(s)y_0(s) = 1 - c_1^2, \\ &x_0(s)^2 + y_0(s)^2 + 2x_0(s)y_0(s) - 6(x_0(s) + y_0(s)) - u_0^2(s) = c_2 \end{aligned}$$

which gives a relation between $c_1$ and $c_2$

$$36(2 - c_1^2) = (1 - c_1^2 - c_2)^2.$$

- And hence, desired surface is

$$36(2 - \phi^2) = (1 - \phi^2 - \psi)^2.$$