MA 542 Differential Equations
Lecture 3
(January 10, 2022)

Integrating Factors

It has been noticed that exact differential equations are rare because exactness depends on a precise balance in the form of the equation and is easily destroyed by minor changes in this form.

Our objective is always to have or convert the given differential equation into an integrable form so as to get the solution.

Consider the following equation:

$$y\;dx + (x^2y - x)\;dy = 0, \tag{1}$$

which is easily seen to be non-exact.

However, if we multiply the whole equation by the factor $1/x^2$, it becomes exact and hence required integration can be carried out to obtain its solution.

In general,

We have to find a proper way, if the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2}$$

is non-exact, then how to make it exact.

In other words,

we have to find a function $\mu(x,y)$ with the property that

$$\mu(M(x,y)\,dx + N(x,y)\,dy) = 0 \tag{3}$$

is exact. Any function $\mu(x,y)$ that acts in this way is called an integrating factor for equation (2).

Thus $1/x^2$ is an integrating factor for (1).

We shall now go ahead to establish that equation (2) always has an integrating factor if it has a general solution.

Assume that (2) has a general solution

$$f(x, y) = c.$$

Eliminate c by differentiating:

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0. \tag{4}$$

It follows from (2) and (4) that

$$\frac{dy}{dx} = -\frac{M}{N} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Subsequently,

$$\frac{\partial f/\partial x}{M} = \frac{\partial f/\partial y}{N}. \tag{5}$$

If we denote the common ratio in (5) by $\mu(x, y)$, then

$$\frac{\partial f}{\partial x} = \mu M \quad \text{and} \quad \frac{\partial f}{\partial y} = \mu N.$$

On multiplying (2) by μ, it becomes

$$\mu M\, dx + \mu N\, dy = 0$$

or

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0,$$

which is exact.

This argument shows that if (2) has a general solution, then it has at least one integrating factor μ.

In general, it is not quite easy to find integrating factors. However, some formal procedures are available with the help of which we can find the integrating factors.

Consider equation (2):

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

Then,

1. If
$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = g(x)$$
then $\mu = e^{\int g(x)\,dx}$ is an integrating factor.

2. If
$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = h(y)$$
then $\mu = e^{\int h(y)\,dy}$ is an integrating factor.

3. If
$$\frac{\partial M/\partial y - \partial N/\partial x}{Ny - Mx} = g(z),\ z = xy$$
then $\mu = e^{\int g(z)\,dz}$ is an integrating factor.

Example: Consider the equation

$$(x^3 + xy^4)\ dx + 2y^3\ dy = 0.$$

Solution:

$$M = x^3 + xy^4, \quad N = 2y^3,$$
$$\frac{\partial M}{\partial y} = 4xy^3, \quad \frac{\partial N}{\partial x} = 0.$$

The integrating factor can be determined from

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{1}{2y^3}(4xy^3) = 2x.$$

The integrating factor can be determined as

$$e^{\int 2x\ dx} = e^{x^2}.$$

The given equation becomes

$$x(x^2 + y^4)e^{x^2}\ dx + 2e^{x^2}y^3\ dy = 0.$$

The solution can be obtained as (in short-cut method)

$$\int x(x^2+y^4)e^{x^2}\, dx + \int 0\, dy = c, \quad \text{(Following: } \int \mu M dx + \int \text{(term containing only } y \text{ in } \mu N\text{)dy.)}$$

Substituting $x^2 = z$:

$$\frac{1}{2}\int (z+y^4)e^z\, dz = c.$$

After integration

$$(z-1+y^4)e^z = 2c,$$

or, the required solution can be written as

$$(x^2-1+y^4)e^{x^2} = 2c.$$

Exercise:

$$(2y\sin x - \cos^3 x)\, dx + \cos x\, dy = 0.$$

Linear equations

The general first-order linear equation can be written as

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{6}$$

The simplest method of solving this depends on the observation that

$$\frac{d}{dx}(e^{\int P\,dx}y) = e^{\int P\,dx}\frac{dy}{dx} + yPe^{\int P\,dx} = e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right). \tag{7}$$

Accordingly, if (6) is multiplied throughout by $e^{\int P\,dx}$, it becomes

$$\frac{d}{dx}(e^{\int P\,dx}y) = Qe^{\int P\,dx}. \tag{8}$$

Integration now leads to

$$e^{\int P\,dx}y = \int Qe^{\int P\,dx}\,dx + c,$$

so that the solution can be obtained as $y = e^{-\int P\,dx}(\int Qe^{\int P\,dx}\,dx + c)$. (9)

Example: Solve

$$(2y + x^2)dx = x\ dy.$$

Solution: The equation can be written as

$$\frac{dy}{dx} - \frac{2}{x}y = x.$$

Here $P(x) = -\frac{2}{x}$ and hence the integrating factor is

$$\text{I.F.} = e^{\int P\ dx} = e^{-\int \frac{2}{x}\ dx} = e^{-2\ln x} = 1/x^2.$$

The solution can be obtained as

$$\begin{aligned} y \times \frac{1}{x^2} &= \int \frac{x}{x^2}\ dx + \ln c \\ &= \ln x + \ln c \\ \Rightarrow y &= x^2 \ln(cx) = x^2 \ln x + Cx^2. \end{aligned}$$

Example:

Find the solution of

$$\frac{dy}{dx} = 3x + y$$

which passes through the point $(-1, 1)$.

Solution:

Here $P(x) = -1$ and hence the integrating factor is

$$\text{I.F.} = e^{\int(-1)\,dx} = e^{-x}.$$

The solution can be obtained as

$$\begin{aligned} y \times e^{-x} &= 3\int xe^{-x}\,dx + c \\ &= 3[-xe^{-x} + \int e^{-x}\,dx] + c \\ &= -3e^{-x}(x+1) + c \\ \Rightarrow y &= -3(x+1) + ce^{x}. \end{aligned}$$

Since the given condition is $y(-1) = 1$:

$$1 = ce^{-1} \Rightarrow c = e.$$

The required solution is

$$y = -3(x+1) + e^{x+1}.$$

Reduction of order:

The most general second-order differential equation has the form

$$F(x, y, y', y'') = 0.$$

Here we consider two special types of second order equations that can be solved by first order methods.

I. Dependent variable absent:

If y is not explicitly present, the above equation can be written in the form

$$F(x, y', y'') = 0. \tag{10}$$

In this case we introduce a new dependent variable p by putting

$$y' = p \quad \text{and} \quad y'' = \frac{dp}{dx}. \tag{11}$$

This substitution transforms (10) into the first order equation

$$F(x, p, \frac{dp}{dx}) = 0. \tag{12}$$

We can solve for p from the equation (12) and then by replacing p with dy/dx, we can solve for y.

This method reduces the problem of solving a second-order equation to solving two first-order equations in succession - one in p and then one in y.

Example:

Solve $xy'' - y' = 3x^2$.

Solution:

By putting

$$y' = p \quad \text{and} \quad y'' = \frac{dp}{dx},$$

The equation becomes

$$xp' - p = 3x^2.$$

The equation gives

$$\frac{dp}{dx} - \frac{1}{x}p = 3x.$$

The integrating factor can be obtained as

$$\text{I.F.} = e^{-\int \frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}.$$

The solution for p can be obtained as

$$\begin{aligned} p \times \frac{1}{x} &= 3\int \frac{x}{x}\, dx + c \\ &= 3x + c \\ \Rightarrow p &= 3x^2 + cx. \end{aligned}$$

Now since $p = y'$, we have another first-order equation:

$$\frac{dy}{dx} = 3x^2 + cx,$$

which can be solved to get

$$y = x^3 + \frac{1}{2}cx^2 + d.$$

II. Independent variable absent:

If x is not explicitly present, the most general second order equation can be written as

$$G(y, y', y^{''}) = 0. \tag{13}$$

Consider

$$y' = p, \quad \text{and} \quad y^{''} = \frac{dp}{dx} = \frac{dp}{dy}\,\frac{dy}{dx} = p\,\frac{dp}{dy}. \tag{14}$$

This enables us to write (13) in the form

$$G\left(y, p, p\frac{dp}{dy}\right) = 0, \tag{15}$$

and proceed to solve two first-order equations in succession - one in p and then one in y.

Example:

Solve $y'' + y = 0$.

Solution:

The equation can be written as

$$p\frac{dp}{dy} + y = 0$$
$$\Rightarrow p\ dp = -y\ dy.$$

Integrating

$$p^2 = c^2 - y^2$$
$$\Rightarrow p = \sqrt{c^2 - y^2}.$$

Since $p = dy/dx$:

we get

$$\frac{dy}{\sqrt{c^2 - y^2}} = dx.$$

Now integrating

$$\sin^{-1}\frac{y}{c} = x + d.$$

This gives us the solution as

$$y = c\sin(x + d).$$