MA 542: Differential Equations
Lecture - 27

March 8, 2022

How do first-order PDEs occur?

- First-order PDEs mainly connect to geometry.
- **Two-parameter family of surfaces:** Let

$$f(x, y, u, a, b) = 0 \tag{1}$$

represent two parameters family of surfaces in $\mathbb{R}^3$, where a and b are arbitrary constants.
Differentiating (1) with respect to x and y yields relations

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u} = 0, \tag{2}$$

$$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial u} = 0. \tag{3}$$

Eliminating a and b from (1), (2) and (3), we get a relation of the form

$$F(x, y, u, p, q) = 0, \tag{4}$$

which is a PDE for the unknown function u of two independent variables x and y.

Example

The equation

$$x^2 + y^2 + (u - c)^2 = r^2, \tag{5}$$

where r and c are arbitrary constants, represents the set of all spheres whose centers lie on the u-axis.

Differentiating (5) with respect to x and y, respectively, we obtain

$$x + (u - c)\frac{\partial u}{\partial x} = 0. \tag{6}$$

$$y + (u - c)\frac{\partial u}{\partial y} = 0. \tag{7}$$

Eliminating the arbitrary constant c from (6) and (7), we obtain a first-order PDE:

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0. \tag{8}$$

Example

The equation

$$x^2 + y^2 = (u - c)^2 \tan^2 \alpha, \tag{9}$$

where c and α are arbitrary constants, represents a family of all right circular cones having u-axis as their axes.
Differentiating (9) with respect to x and y, respectively, we obtain

$$\frac{\partial u}{\partial x}(u - c) \tan^2 \alpha = x. \tag{10}$$

$$\frac{\partial u}{\partial y}(u - c) \tan^2 \alpha = y. \tag{11}$$

Eliminating the arbitrary constants c and α from (10) and (11), we obtain a first-order PDE:

$$y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0, \tag{12}$$

which is interestingly the same as (8).

- **Unknown function of known functions**
  - *Unknown function of a single known function*

    Let

    $$u = f(g), \tag{13}$$

    where f is an unknown function and g is a known function of two independent variables x and y.

    Differentiating (13) with respect to x and y, respectively, yields the equations

    $$u_x = f'(g)g_x \tag{14}$$

    and

    $$u_y = f'(g)g_y. \tag{15}$$

    Eliminating $f'(g)$ from (17) and (18), we obtain

    $$g_y u_x - g_x u_y = 0,$$

    which is a first-order PDE for u.

Example

Consider the surfaces described by an equation of the form

$$u = f(x^2 + y^2), \tag{16}$$

where f is an arbitrary function of a known function $g(x, y) = x^2 + y^2$. Differentiating (16) with respect to x and y, it follows that

$$u_x = 2xf'(g), \qquad u_y = 2yf'(g),$$

where $f'(g) = \frac{df}{dg}$.
Eliminating $f'(g)$ from the above two equations, we obtain a first-order PDE

$$yu_x - xu_y = 0.$$

The forms of (5) and (9) allow for both to correspond to the same PDE. That is, both (5) and (9) have the similar properties as mentioned, i.e., $u = f(x^2 + y^2)$.

- *Unknown functions of two known functions*
  Let
  $$u = f(x - ay) + g(x + ay), \tag{17}$$
  where $a > 0$ is a constant.
  With $v(x, y) = x - ay$ and $w(x, y) = x + ay$, we can write (17) as
  $$u = f(v) + g(w). \tag{18}$$
  Differentiating (18) w. r. t. x and y, respectively, yields
  $$\begin{aligned} p = u_x &= f'(x - ay) + g'(x + ay), \\ q = u_y &= -af'(x - ay) + ag'(x + ay). \end{aligned}$$
  Eliminating $f'(v)$ and $g'(w)$ (by differentiating above again), we get
  $$q_y = a^2 p_x.$$
  In terms of u, the above PDE is the well-known one-dimensional wave equation
  $$u_{yy} = a^2 u_{xx}.$$

Example (Geometrical problem)

All functions $u(x, y)$ such that the tangent plane to the graph $u = u(x, y)$ at any arbitrary point $(x_0, y_0, u(x_0, y_0))$ passes through the origin is characterized by the PDE $xu_x + yu_y - u = 0$.

The equation of the tangent plane at $(x_0, y_0, u(x_0, y_0))$ is

$$u_x(x_0, y_0)(x - x_0) + u_y(x_0, y_0)(y - y_0) - (u - u(x_0, y_0)) = 0.$$

Since this plane passes through the origin $(0, 0, 0)$, we have

$$-u_x(x_0, y_0)x_0 - u_y(x_0, y_0)y_0 + u(x_0, y_0) = 0. \tag{19}$$

For equation (19) to hold for all (x_0, y_0) in the domain of u, we must have u satisfying

$$xu_x + yu_y - u = 0,$$

which is a first-order PDE.

Cauchy's problem or IVP for first-order PDEs:

Let Γ be a given curve in $\mathbb{R}^2$ described parametrically by the equations

$$x = x_0(s), \qquad y = y_0(s); \quad s \in I, \tag{20}$$

where $x_0(s)$, $y_0(s)$ are in $C^1(I)$.

The IVP or Cauchy's problem for the first-order PDE

$$F(x, y, u, p, q) = 0 \tag{21}$$

is to find a function $\phi = \phi(x, y)$ with the following properties:

- $\phi(x, y)$ and its partial derivatives with respect to x and y are continuous in a region Ω of $\mathbb{R}^2$ containing the curve Γ.
- $\phi = \phi(x, y)$ is a solution of (21) in Ω, i.e.,
$$F(x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)) = 0 \quad \text{in } \Omega.$$
- On the curve Γ,
$$\phi(x_0(s), y_0(s)) = u_0(s), \quad s \in I. \tag{22}$$

The curve Γ is called the initial curve of the problem and the function $u_0(s)$ is called the initial data. Equation (22) is called the initial condition (or side condition) of the problem.

- **Well-posed Problem** (In the sense of Hadamard)

The Cauchy's problem (PDE + side condition) is said to be well-posed if it satisfies the following criteria:

1. The solution exists.
2. The solution is unique.
3. The solution depends continuously on the initial and/or boundary data.

If one or more of the above conditions does not/do not hold, we say that the problem is ill-posed.

Linear First-Order PDEs

The most general first-order linear PDE has the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y), \tag{23}$$

where a, b, c, and d are given functions of x and y. These functions are assumed to be continuously differentiable.
Observe that the left hand side of (23), i.e.,

$$a(x,y)u_x + b(x,y)u_y = \nabla u \cdot (a,b)$$

is (essentially) a directional derivative of $u(x,y)$ in the direction of the vector (a,b), where (a,b) is defined and nonzero.

Remarks: When a and b are constants, the vector (a,b) has a fixed direction and magnitude, but now it is seen that the vector (a,b) can change as its base point (x,y) varies. Thus, (a,b) is a vector field on the plane.

The equations

$$\boxed{\frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y),} \tag{24}$$

determine a family of curves $x = x(t)$, $y = y(t)$ whose tangent vector $(\frac{dx}{dt}, \frac{dy}{dt})$ coincides with the direction of the vector (a, b).
Therefore,

$$\begin{aligned} \frac{d}{dt}u\{(x(t),y(t))\} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\ &= u_x(x(t),y(t))a(x(t),y(t)) \\ &\quad +u_y(x(t),y(t))b(x(t),y(t)) \\ &= c(x(t),y(t))u(x(t),y(t)) + d(x(t),y(t)) \\ &= c(t)u(t) + d(t), \end{aligned}$$

where we have used the chain rule and (23).
Thus, along these curves, $u(t) = u(x(t),y(t))$ satisfies the ODE

$$u'(t) - c(t)u(t) = d(t). \tag{25}$$

Let $\mu(t) = \exp\left[-\int_0^t c(\tau)d\tau\right]$ be an integrating factor for (25). Then, the solution is given by

$$u(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(\tau)d(\tau)d\tau + u(0)\right]. \tag{26}$$

The approach described above is called the method of characteristics. It is based on the geometric interpretation of PDE (23).

Remarks.

- The system of ODEs (24) is known as the characteristic equation for the PDE (23). The solution curves of the characteristic equation are the characteristic curves for (23).
- The values $u(t)$ of the solution u along the entire characteristic curves can be completely determined once the value $u(0) = u(x(0), y(0))$ is prescribed.
- Assuming certain smoothness conditions on the functions a, b, c, and d, the existence and uniqueness theory for ODEs guarantees a unique solution curve $(x(t), y(t), u(t))$ of (24) and (25).