MA 542: Differential Equations
Lecture - 26

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Definition

A partial differential equation (PDE) for a function $u(x_1, x_2, \ldots, x_n)$ $(n \geq 2)$ is a relation of the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_1 x_1}, u_{x_1 x_2}, \ldots,) = 0, \qquad (1)$$

where F is a given function of the independent variables $x_1, x_2, \ldots, x_n$; of the unknown function u and of a finite number of its partial derivatives.

Definition (Solution of a PDE)

A function $\phi(x_1, \ldots, x_n)$ is a solution to (1) if ϕ and its partial derivatives appearing in (1) satisfy (1) identically for $x_1, \ldots, x_n$ in some region $\Omega \subset \mathbb{R}^n$.

The order of an equation:The order of a PDE is the order of the highest derivative appearing in the equation. If the highest derivative is of order m, then the equation is said to be order m.

$$u_t - u_{xx} = f(x, t) \ \text{(second-order equation)}$$

$$u_t + u_{xxx} + u_{xxxx} = 0 \ \text{(fourth-order equation)}$$

Definition (Classification)

- A PDE is said to be linear if it is linear in the unknown function u and its partial derivatives, with the coefficients depending only on the independent variables $x_1, x_2, \ldots, x_n$.
- A PDE of order m is said to be quasi-linear if it is linear in the derivatives of order m with coefficients that depend on $x_1, x_2, \ldots, x_n, u$ and the derivatives of order $< m$.
- A quasi-linear PDE of order m, where the coefficients of derivatives of order m are functions of the independent variables $x_1, \ldots, x_n$ alone is called a semi-linear PDE.
- A PDE of order m is called nonlinear if it is not linear in the derivatives of order m.

Example (Some well-known PDEs)

- The Laplace's equation in n dimensions:

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \text{ (second-order, linear, homogeneous)}$$

- The Poisson equation:

$$\Delta u = f \text{ (second-order, linear, nonhomogeneous)}$$

- The heat conduction (diffusion) equation:

$$\frac{\partial u}{\partial t} - k\Delta u = 0 \;\; (k = \text{const.} > 0) \text{ (second-order, linear, homogeneous)}$$

- The wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \;\; (c = \text{const.} > 0) \text{ (second-order, linear, homogeneous)}$$

- The Transport equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \text{ (first-order, linear, homogeneous)}$$

- The Burger's equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \text{ (first-order, quasilinear, homogeneous)}$$

Basic facts about ODE and PDE:

- Let $u = u(x, y)$ and consider a PDE $u_x = \frac{\partial u}{\partial x} = 0$. Integrating it, we have $u = u(x, y) = c(y)$, i.e., any arbitrary function of y solves this PDE.
- Solution $u(x, y) = c(y)$ gives all possible solutions of the PDE. Such a solution is called a general solution/integral.
- In PDE, a general solution involves arbitrary functions, whereas in ODE, a general solution involves arbitrary constants only.

What type of equations are of interest?

- Linear, quasi-linear, and nonlinear first-order PDEs involving two independent variables.
- Linear second-order PDEs in two/three independent variables.

Example

Let us warm up with a simple example

$$u_x = u + c, \quad c \text{ is function of } \ x, \ y. \tag{2}$$

Observe

- Since equation (2) contains no derivative with respect to the variable y, we can regard this variable as a parameter.
- Thus, for fixed y, we are actually dealing with an ODE, the solution is immediate:

$$u(x,y) = e^x \Big[\int_0^x e^{-\xi} c(\xi, y) d\xi + f_1(y) \Big]. \tag{3}$$

- Suppose, we supplement (2) with the initial condition $u(0,y) = y$.
- Then the unique solution is given by

$$u(x,y) = e^x \Big[\int_0^x e^{-\xi} c(\xi, y) d\xi + y \Big]. \tag{4}$$

Example

- Consider following IVP

$$u_x = u, \quad u(x,0) = 2x. \tag{5}$$

- The solution of (5) now becomes $u(x,y) = e^x f_2(y)$ and with the condition $u(x,0) = 2x$, we must have $f_2(0) = 2xe^{-x}$, which is of course impossible.
- We have seen so far an example in which a problem had a unique solution,and another example where there was no solution at all. It turns out that an equation might have infinitely many solutions.
- Consider following IVP

$$u_x = u, \quad u(x,0) = e^x. \tag{6}$$

- Now $f_2(y)$ should satisfy $f_2(0) = 1$. Thus every function $f_2(y)$ satisfying $f_2(0) = 1$ will provide a solution for the equation together with the initial condition. Hence, the IVP has infinitely many solutions.

First-order PDEs: A first-order PDE in two independent variables x, y and the dependent variable u can be written in the form

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0. \tag{7}$$

For convenience, set

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}.$$

Equation (7) then takes the form

$$F(x, y, u, p, q) = 0. \tag{8}$$

First-order PDEs arise in many applications, such as

- Transport of material in a fluid flow.
- Propagation of wave-fronts in optics.

- **Classification of first-order PDEs**
  - If (7) is of the form

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)u + d(x,y),$$

    then it is called a linear first-order PDE.
  - If (7) has the form

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u),$$

    then it is called a semilinear PDE because it is linear in the leading (highest-order) terms $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. However, it need not be linear in u.
  - If (7) has the form

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u),$$

    then it is called a quasi-linear PDE. Here the function F is linear in the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ with the coefficients a, b and c depending on the independent variables x and y as well as on the unknown u.
  - If F is not linear in the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, then (7) is said to be a nonlinear PDE.

Linear PDE $\subsetneq$ Semi-linear PDE $\subsetneq$ Quasi-linear PDE $\subsetneq$ PDE

Examples

- $xu_x + yu_y = u$ (**linear**)
- $xu_x + yu_y = u^2$ (**semi-linear**)
- $u_x + (x+y)u_y = xy$ (**linear**)
- $uu_x + u_y = 0$ (**quasi-linear**)
- $xu_x^2 + yu_y^2 = 2$ (**nonlinear**)

How do first-order PDEs occur?

- First-order PDEs mainly connect to geometry.
- **Two-parameter family of surfaces:** Let

$$f(x, y, u, a, b) = 0 \tag{9}$$

represent two parameters family of surfaces in $\mathbb{R}^3$, where a and b are arbitrary constants.
Differentiating (9) with respect to x and y yields a relations

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u} = 0, \tag{10}$$

$$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial u} = 0. \tag{11}$$

Eliminating a and b from (9), (10) and (11), we get a relation of the form

$$F(x, y, u, p, q) = 0, \tag{12}$$

which is a PDE for the unknown function u of two independent variables x and y.