MA 542 Differential Equations

Lecture 24

(February 22, 2022)

To start with:

- The approach towards a solution of a specific BVP can be made in many ways.
- Here we are going to discuss a particular method which requires the construction of an auxiliary function known as Green's function.
- We begin with a typical one-dimensional BVP, i.e., a BVP governed by an ordinary differential equation.

Consider the problem of the forced, transverse vibrations of a taut string of length L. This is in general an evolutionary equation depending on the independent variables x and t. However, if the time-dependent part of the equation is removed (by the separation of variables technique), we obtain the following ordinary differential equation containing the transverse displacement of the string, u, as the unknown and depending only on x:

$$\frac{d^2u}{dx^2} + k^2u = -f(x), \; 0 < x < L. \tag{1}$$

If the ends of the string $x = 0$ and $x = L$ are kept tied, then this equation must be solved for u subject to the boundary conditions:

$$u(0) = u(L) = 0. \tag{2}$$

To solve the BVP posed by the differential equation (1) and associated boundary conditions (2), we employ the method of variation of parameters.

The solution can be assumed to be of the form:

$$u(x) = A(x)\cos kx + B(x)\sin kx, \tag{3}$$

since the solution of the homogenous equation (when $f(x) \equiv 0$) is $u(x) = A\cos kx + B\sin kx$.

Then

$$\begin{aligned} A'(x)\cos kx + B'(x)\sin kx &= 0, & (4)\\ -kA'(x)\sin kx + kB'(x)\cos kx &= -f(x). & (5) \end{aligned}$$

Now solving the above equations for $A'(x)$ and $B'(x)$, we get

$$A'(x) = \frac{f(x)\sin kx}{k}, \quad B'(x) = \frac{-f(x)\cos kx}{k}. \tag{6}$$

Integrating the above with respect to x, we get

$$A(x) = \int_{c_1}^{x} \frac{f(y)\sin ky}{k} dy, \tag{7}$$

$$B(x) = -\int_{c_2}^{x} \frac{f(x)\cos ky}{k} dy, \tag{8}$$

where c_1 and c_2 are constants which must be so chosen as to ensure that the boundary conditions (2) are satisfied.

Thus now we can now write the solution of (1) as

$$u(x) = \frac{\cos kx}{k} \int_{c_1}^{x} f(y)\sin ky \; dy - \frac{\sin kx}{k} \int_{c_2}^{x} f(y)\cos ky \; dy. \tag{9}$$

Upon utilizing the first boundary condition $u(0) = 0$, we get

$$\frac{1}{k}\int_{c_1}^{0} f(y)\sin ky \; dy = 0,$$
$$\Rightarrow c_1 = 0 \quad (\text{since } f(y) \text{ is arbitrary}).$$

Thus now we can now write the solution of (1) as

$$u(x) = \frac{\cos kx}{k}\int_{0}^{x} f(y)\sin ky \; dy - \frac{\sin kx}{k}\int_{c_2}^{x} f(y)\cos ky \; dy. \tag{10}$$

Now utilizing the second boundary condition $u(L) = 0$, we get

$$0 = \frac{\cos kL}{k}\int_{0}^{L} f(y)\sin ky \; dy - \frac{\sin kL}{k}\int_{c_2}^{L} f(y)\cos ky \; dy. \tag{11}$$

From (11)

$$\int_{c_2}^{L} f(y)\cos ky \; dy = \frac{\cos kL}{\sin kL}\int_{0}^{L} f(y)\sin ky. \tag{12}$$

That is

$$\int_0^L f(y)\cos ky\ dy - \int_0^{c_2} f(y)\cos ky\ dy = \frac{\cos kL}{\sin kL}\int_0^L f(y)\sin ky$$

$$\Rightarrow \int_0^{c_2} f(y)\cos ky\ dy = \int_0^L f(y)\cos ky\ dy - \frac{\cos kL}{\sin kL}\int_0^L f(y)\sin ky$$

$$\Rightarrow \int_0^{c_2} f(y)\cos ky\ dy = \frac{1}{\sin kL}\int_0^L \sin k(L-y)f(y)\ dy.$$

Going back to (10) :

$$\begin{aligned}
u(x) &= \frac{\cos kx}{k}\int_0^x f(y)\sin ky\ dy - \frac{\sin kx}{k}\int_{c_2}^x f(y)\cos ky\ dy \\
&= \frac{\cos kx}{k}\int_0^x f(y)\sin ky\ dy - \frac{\sin kx}{k}\left\{\int_0^L f(y)\cos ky\ dy\right. \\
&- \left.\int_x^L f(y)\cos ky\ dy - \int_0^{c_2} f(y)\cos ky\ dy\right\} \\
&= \frac{\cos kx}{k}\int_0^x f(y)\sin ky\ dy - \frac{\sin kx}{k}\left\{\int_0^L f(y)\cos ky\ dy\right. \\
&- \left.\int_x^L f(y)\cos ky\ dy - (1/\sin kL)\int_0^L \sin k(L-y)f(y)\ dy.\right\}
\end{aligned} \tag{13}$$

Now the terms in (13) can be adjusted to write as

$$
\begin{aligned}
u(x) &= \int_0^x \frac{\sin ky \sin k(L-x)}{k \sin kL} f(y)dy + \int_x^L \frac{\sin kx \sin k(L-y)}{k \sin kL} f(y)dy \\
&= \int_0^L G(x,y) f(y) dy, \qquad (14)
\end{aligned}
$$

where

$G(x, y)$ is defined as

$$
G(x,y) = \begin{cases} \dfrac{\sin ky \sin k(L-x)}{k \sin kL}; & 0 \leq y < x, \\[2ex] \dfrac{\sin kx \sin k(L-y)}{k \sin kL}; & x < y \leq L. \end{cases} \qquad (15)
$$

This function, $G(x, y)$, is a two-point function of position known as the Green's function for equation (1) subject to boundary conditions (2).

The existence of $G(x, y)$ and hence the solution is assured for this particular problem, provided that $\sin kL \neq 0$. Therefore, when $G(x, y)$ exists and when it is known explicitly, we can immediately write down the solution to our BVP consisting of (1) and (2) in the particularly simple form (14).

Advantage 1

$G(x, y)$ is independent of the forcing term $f(x)$ and depends only upon the particular differential equation being examined and the boundary conditions which are imposed.

Advantage 2

The integral representation (14) is much more amenable to numerical analysis than is the original differential equation and associated boundary conditions.

Notations

Green's function is also denoted in many books as $G(x; \xi)$ instead of $G(x, y)$. We have used x and y for easy understanding.
Similarly depending on the type of the BVPs, the Green's functions may have notations like $G(x, y; \xi, \eta)$ or $G(x, y, z; \xi, \eta, \zeta)$, both of which will appear when encountering BVPs governed by PDEs.

Properties

• It satisfies the homogeneous form of the given differential equation, that is

$$G'' + k^2 G = 0$$

in each of the intervals $0 \leq y < x,\ x < y \leq L$. The behaviour of G at $y = x$ is, at this moment, uncertain.

• The function G is continuous at $y = x$ since

$$\lim_{x \to y^-} G(x,y) = \frac{\sin kx \sin k(L-x)}{k \sin kL} = \lim_{x \to y^+} G(x,y).$$

• The derivative of G with respect to y is discontinuous at $y = x$. This can be seen from the following:

$$\begin{aligned} G'(x,y^-) &= \lim_{x \to y^-} G'(x,y) = \frac{\cos kx \sin k(L-x)}{\sin kL} \\ G'(x,y^+) &= \lim_{x \to y^+} G'(x,y) = \frac{-\sin kx \cos k(L-x)}{\sin kL} \end{aligned}$$

Hence

$$G'(x,y^+) - G'(x,y^-) = -1.$$

Properties

- The function $G(x, y)$ satisfies the relations

$$G(x, 0) = 0,\ \ G(x, L) = 0.$$

- The function $G(x, y)$ is symmetric in its arguments. Hence

$$G(x, y) = G(y, x).$$

Remark

With these several properties of the Green's function in mind, we can try to solve the given BVP, given by equations (1) and (2), by assuming from the outset the existence of a Green's function, $G(x, y)$.

Remark

If such an assumption is valid, we should be able to recover directly from the differential equation, not only the representation (14) but also several properties of $G(x, y)$.

Solving a Problem

We have demonstrated a problem in which the solution could be expressed in terms of a Green's function. The process will be reversed if we want to construct a Green's function and hence find the solution.

We will be required to use the properties to find a Green's function for the specific problem.

Applications of these properties will allow us to eliminate the coefficients in the general solution and find an expression in each of the intervals $a < x < y$ and $y < x < b$ over an interval $a < x < b$.

Now let us try solving the same problem as discussed earlier but now by using the properties of Green's function.

We know that the solution will be given by

$$u(x) = \int_0^L G(x,\xi) f(\xi) d\xi.$$

We are required to find $G(x,\xi)$.

Since $G(x,\xi)$ satisfies the homogeneous equation

$$\frac{d^2 G}{dx^2} + k^2 G = 0, \tag{16}$$

in each of the intervals $0 \leq \xi < x$ and $x < \xi \leq L$,

we can write $G(x,\xi)$ as

$$G(x,\xi) = A(\xi) \cos kx + B(\xi) \sin kx. \tag{17}$$

We will evaluate A and B by using various properties of $G(x,\xi)$.

Using the boundary conditions

$$G(x,\xi) = \begin{cases} B(\xi)\sin kx, & 0 \le \xi < x \\ A(\xi)\frac{\sin k(L-x)}{\sin kL}, & x < \xi \le L. \end{cases} \tag{18}$$

Using the continuity condition of $G(x,\xi)$ at $x = \xi$:

$$G(x,\xi^-) = G(x,\xi^+) \tag{19}$$

$$\Rightarrow A(\xi)\frac{\sin k(L-\xi)}{\sin kL} - B(\xi)\sin k\xi = 0. \tag{20}$$

Using $G'(x,\xi^+) - G'(x,\xi^-) = -1$:

$$A(\xi)\frac{k\cos k(L-\xi)}{\sin kL} + B(\xi)k\cos k\xi = 1. \tag{21}$$

$A(\xi)$ and $B(\xi)$ can be evaluated from (20) and (21) as

$$A(\xi) = \frac{\sin k\xi \sin kL}{k}, \tag{22}$$

$$B(\xi) = \frac{\sin k(L-\xi)}{k \sin kL}. \tag{23}$$

Using (22) and (23) in (18)

$$G(x,\xi) = \begin{cases} \frac{\sin k(L-\xi)}{k \sin kL} \sin kx, & 0 \leq \xi < x, \\ \sin k\xi \frac{\sin k(L-x)}{k \sin kL}, & x < \xi \leq L. \end{cases} \tag{24}$$

We can now write the solution $u(x)$ as

$$u(x) = \int_0^L G(x,\xi) f(\xi) d\xi. \tag{25}$$