MA 542 Differential Equations

Lecture 23

(February 21, 2022)

Stability by Lypunov's Direct Method

Observe the following regarding an equilibrium point of a conservative dynamical system: If the potential energy has a relative minimum at the equilibrium point, then the equilibrium point is stable; otherwise it is unstable. This principle was generalized by the Russian mathematician Lypunov (or Liapunov) to obtain a simple but powerful method for studying the stability of more general autonomous systems. The procedure is known as Lypunov's direct (or second) method.

Consider an autonomous system

$$\left.\begin{array}{l} \frac{dx}{dt} = F(x,y), \\ \frac{dy}{dt} = G(x,y). \end{array}\right\} \qquad (1)$$

Assume that this system has an isolated critical point at the origin $(0,0)$ and that F and G have continuous first-order partial derivatives for all (x,y).

Let $C = [x(t), y(t)]$ be a path of (1) and consider a function $E(x,y)$ that is continuous and has continuous first-order partial derivatives in a region containing this path.

If a point (x, y) moves along the path in accordance with the equations $x = x(t)$ and $y = y(t)$, then $E(x, y)$ can be regarded as a function of t along C, and its rate of change is

$$\begin{aligned} \frac{dE}{dt} &= \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G. \end{aligned} \tag{2}$$

Some definitions

Suppose that $E(x, y)$ is continuous and has continuous first-order partial derivatives at all points (x, y) in some region D containing the origin.

1. The function $E(x, y)$ is called positive definite in D if $E(0, 0) = 0$ and $E(x, y) > 0$ for all other points $(x, y) \neq (0, 0)$ in D.
2. The function $E(x, y)$ is called positive semidefinite in D if $E(0, 0) = 0$ and $E(x, y) \geq 0$ for all other points $(x, y) \neq (0, 0)$ in D.
3. The function $E(x, y)$ is called negative definite in D if $E(0, 0) = 0$ and $E(x, y) < 0$ for all other points $(x, y) \neq (0, 0)$ in D.
4. The function $E(x, y)$ is called negative semidefinite in D if $E(0, 0) = 0$ and $E(x, y) \leq 0$ for all other points $(x, y) \neq (0, 0)$ in D.

Observations

- It is clear that functions of the form $E(x, y) = ax^{2m} + by^{2n}$, where a and b are positive constants and m and n are positive integers, are positive definite.
- Since $E(x, y)$ is negative if and only if $-E(x, y)$ is positive definite, functions of the form $ax^{2m} + by^{2n}$ with $a < 0$ and $b < 0$ are negative definite.
- The functions x^{2m}, y^{2n} and $(x - y)^{2m}$ are not positive definite, but are nevertheless positive semidefinite.
- If $E(x, y)$ is positive definite, then $z = E(x, y)$ can be interpreted as the equation of a surface that resembles a paraboloid opening upward and tangent to the xy-plane at the origin. (Figure 1)

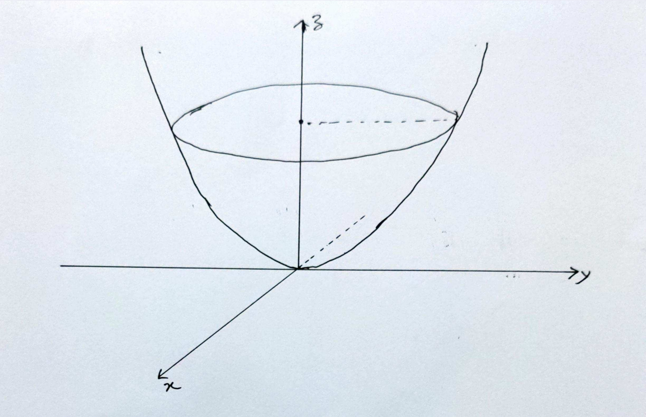

Figure 1

Definition

A positive definite function $E(x, y)$ with the property that

$$\frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G \tag{3}$$

is negative semidefinite is called a Lypunov function of the system (1).

By formula (2), the requirement that (3) be negative semidefinite means that $dE/dt \leq 0$, and therefore E is non-increasing, along with the paths of (1) near the origin. These functions generalize the concept of the total energy of a physical system.

Theorem

If there exists a Lypunov function $E(x, y)$ for the system (1), then the critical point $(0, 0)$ is stable. Furthermore, if this function has the additional property that the function (3) is negative definite, then the critical point $(0, 0)$ is asymptotically stable.

Proof

Let C_1 be a circle of radius $R > 0$ centered on the origin and assume also that C_1 is small enough to lie entirely in the domain of definition of the function E.

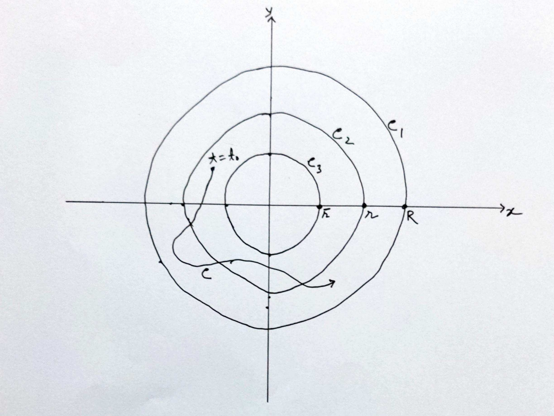

Figure 2

Stability of linear systems

Since $E(x, y)$ is continuous and positive definite, it has a positive minimum m on C_1. Next $E(x, y)$ is continuous at the origin and vanishes there, so we can find a positive number $r < R$ such that $E(x, y) < m$ whenever (x, y) is inside the circle C_2 of radius r.

Now let $C = [x(t), y(t)]$ be any path which is inside C_2 for $t = t_0$. Then $E(t_0) < m$, and since (3) is negative semidefinite, we have $dE/dt \leq 0$, which implies that $E(t) \leq E(t_0) < m$ for all $t > t_0$.

It follows that the path C can never reach the circle C_1 for any $t > t_0$, so we have stability.

To prove the second part of the theorem, it suffices to show that under the additional hypothesis, we also have $E(t) \to 0$, for since $E(x, y)$ is positive definite this will imply that the path C approaches the critical point $(0, 0)$.

We begin by observing that since $dE/dt < 0$, it follows that $E(t)$ is a decreasing function; and since by hypothesis $E(t)$ is bounded below by 0, we conclude that $E(t)$ approaches some limit $L \geq 0$ as $t \to \infty$.

To prove that $E(t) \to 0$, it is sufficient to show that $L = 0$. Therefore, we assume that $L > 0$ and deduce a contradiction. Choose a positive number $\bar{r} < r$ with the property that $E(x, y) < L$ whenever (x, y) is inside the circle C_3 with radius $\bar{r}$.

Since the function (3) is continuous and negative definite, it has a negative maximum $-k$ in the consisting of the circles C_1 and C_3 and the region between them.

This ring contains the entire path C for $t \geq t_0$ and so the equation

$$E(t) = E(t_0) + \int_{t_0}^{t} \frac{dE}{dt}\, dt$$

yields the inequality

$$E(t) \leq E(t_0) - k(t - t_0) \tag{4}$$

for all $t \geq t_0$.

However, the right side of (4) becomes negatively infinite as $t \to \infty$ and therefore $E(t) \to -\infty$ as $t \to \infty$. This contradicts the fact that $E(x, y) \geq 0$ and therefore, we are in a position to conclude that $L = 0$ and the proof is complete.

Example

Consider the equation of a mass m attached to a spring:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0, \tag{5}$$

where $c \geq 0$ is a constant representing the viscosity of the medium through which the mass moves, and $k > 0$ is the spring constant.

Solution

The autonomous system equivalent to (5) is

$$\left.\begin{array}{l} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -\frac{k}{m}x - \frac{c}{m}y, \end{array}\right\} \tag{6}$$

and its only critical point is $(0, 0)$.

The kinetic energy of the mass is $my^2/2$, and the potential energy (or the energy stored in the spring) is

$$\int_0^x k\xi \, d\xi = \frac{1}{2}kx^2.$$

Thus the total energy of the system is

$$E(x,y) = \frac{1}{2}\, my^2 + \frac{1}{2}\, kx^2. \tag{7}$$

It is easy to see that (7) is positive definite and also

$$\frac{\partial E}{\partial x}\, F + \frac{\partial E}{\partial y}\, G = kxy + my\left(-\frac{k}{m}x - \frac{c}{m}y\right) = -cy^2 \leq 0.$$

Based on the above, we observe that (7) is a Lypunov function for (6) and hence the critical point $(0,0)$ is stable.

Example

The system

$$\left.\begin{aligned} \frac{dx}{dt} &= -2xy, \\ \frac{dy}{dt} &= x^2 - y^2, \end{aligned}\right\} \tag{8}$$

has $(0,0)$ as an isolated critical point.

To establish stability, construct a Lypunov function of the form $E(x,y) = ax^{2m} + by^{2n}$. It is clear that

$$\begin{aligned} \frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G &= 2max^{2m-1}(-2xy) + 2nby^{2n-1}(x^2 - y^2) \\ &= -4max^{2m}y + 2nbx^2y^{2n-1} - 2nby^{2n+2}. \end{aligned}$$

By choosing $m = 1, n = 1, a = 1$ and $b = 2$, we have $E(x,y) = x^2 + 2y^2$ which is positive definite, and $(\partial E/\partial x)F + (\partial E/\partial y)G = -4y^4$ which is negative semidefinite. The critical point $(0,0)$ of the system (8) is therefore stable.

However, it may be observed that it may not be easy to construct Lypunov function for complicated situations. In this context, the following theorem may be useful.

Theorem

The function $E(x, y) = ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$, and is negative definite if and only if $a < 0$ and $b^2 - 4ac < 0$.

Proof

If $y = 0$, we have $E(x, 0) = ax^2$, so $E(x, 0) > 0$ for $x \neq 0$ if and only if $a > 0$.

If $y \neq 0$, we have

$$E(x, y) = y^2 \left[a \left(\frac{x}{y} \right)^2 + b \left(\frac{x}{y} \right) + c \right];$$

and when $a > 0$, the bracketed polynomial in x/y (which is positive for large x/y) is positive for all x/y if and only if $b^2 - 4ac < 0$.

This proves the first part of the theorem and the second part follows by considering the function $-E(x, y)$.