MA 542 Differential Equations

Lecture 20

(February 15, 2022)

Types of critical points

Consider the following autonomous system:

$$\left.\begin{aligned} \frac{dx}{dt} &= F(x,y), \\ \frac{dy}{dt} &= G(x,y), \end{aligned}\right\} \tag{1}$$

where the functions F and G are continuous and have continuous first-order partial derivatives throughout the xy-plane. The critical points can be found by solving $F(x,y) = 0$ and $G(x,y) = 0$. There are four simple types of critical points. But before discussing the critical points, we need to know certain other aspects as follows.

Definition: Let (x_0, y_0) be an isolated critical point of (1). If $\Gamma \equiv [x(t), y(t)]$ is a path of (1), then we say Γ approaches (x_0, y_0) as $t \to \infty$ if

$$\lim_{t\to\infty} x(t) = x_0, \quad \lim_{t\to\infty} y(t) = y_0. \tag{2}$$

Geometrically, this means that if $P = (x,y)$ is a point that traces out Γ in accordance with the equations $x = x(t)$ and $y = y(t)$, then $P \to (x_0, y_0)$ as $t \to \infty$.

Definition: If it is also true that

$$\lim_{t \to \infty} \frac{y(t) - y_0}{x(t) - x_0} \tag{3}$$

exists or if the quotient in (3) becomes either positively or negatively infinite as $t \to \infty$, then we say that Γ enters the critical point (x_0, y_0) as $t \to \infty$.

The quotient in (3) is nothing but the slope of the line joining (x_0, y_0) and the point P with coordinates $x(t)$ and $y(t)$.

We may also consider limits as $t \to -\infty$. It is clear that these properties are properties of the path Γ and do not depend on which solution is used to represent the path.

For some cases, it is possible to find the explicit solutions of the system (1) and these solutions can be used to determine the paths. However, in most cases, in order to find the paths, it is necessary to eliminate t between the two constituent equations of the system to get

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}. \tag{4}$$

This first-order equation (4) gives the slope of the tangent to the path of (1) that passes through the point (x, y) provided that the functions $F(x, y)$ and $G(x, y)$ are not both zero at this point.

In this case, of course, this point is a critical point and no path passes through it. The paths of (1) therefore coincide with the one-parameter family of orthogonal curves of (4) and this family can be obtained without much difficulty. However, it may be noted that while the paths of (1) are directed curves, the integral curves of (4) have no direction associated with them.

We now discuss different types of critical points mainly with respect to geometrical interpretation. In most of the cases, the origin $O = (0, 0)$ is the point which is a critical point.

Node

It is a critical point approached and also entered by each path as $t \to \infty$ (or as $t \to -\infty$). Here there are four half-line paths, AO, BO, CO and DO which together with the origin make up the lines AB and CD. All other paths resemble paths of parabolas and as each of these paths approaches O, its slope approaches that of the line AB. Refer to Figure 1 below.

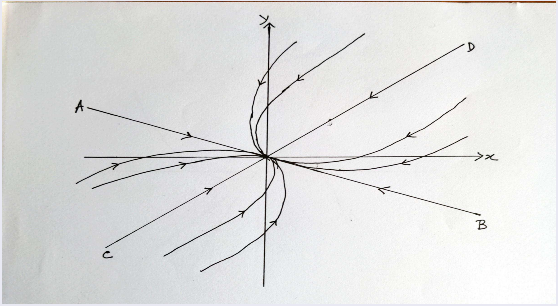


Figure 1

Example: Consider the system

$$\left.\begin{aligned} \frac{dx}{dt} &= x, \\ \frac{dy}{dt} &= -x + 2y. \end{aligned}\right\} \tag{5}$$

Here origin is the only critical point.

Solution

Here $a_1 = 1, b_1 = 0, a_2 = -1, b_2 = 2$ and hence the auxiliary equation $m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0$ for this system is $m^2 - 3m + 2 = 0$ giving us $m = 1, 2$.

Hence the general solution is

$$\left.\begin{aligned} x &= c_1 A_1 e^t + c_2 B_1 e^{2t} = c_1 e^t, \\ y &= c_1 A_2 e^t + c_2 B_2 e^{2t} = c_1 e^t + c_2 e^{2t}. \end{aligned}\right\} \tag{6}$$

When $c_1 = 0$, then $x = 0, y = c_2 e^{2t}$. In this case, the path is the positive y-axis when $c_2 > 0$ and the negative y-axis when $c_2 < 0$. Each path approaches and enters the origin as $t \to -\infty$.

When $c_2 = 0$, then $x = c_1 e^t$, $y = c_1 e^t$. This path is the half-line $y = x, x > 0$ when $c_1 > 0$, and the half-line $y = x, x < 0$ when $c_1 < 0$. Again both paths approach and enter the origin as $t \to -\infty$.

When both c_1 and c_2 are not zero, the paths lie on the parabolas $y = x + (c_2/c_1^2)x^2$, which go through the origin with slope 1. It should be understood that each of these paths consists of only part of a parabola, the part with $x > 0$ if $c_1 > 0$, and the part with $x < 0$ if $c_1 < 0$.
Each of these paths also approaches and enters the origin as $t \to -\infty$; this can be seen at once from (6).

We can also proceed directly from (5):

$$\frac{dy}{dx} = \frac{-x + 2y}{x}, \tag{7}$$

which gives the slope of the tangent to the path through (x, y) [provided $(x, y) \neq (0, 0)$], then on solving (7), we find $y = x + cx^2$.

This procedure yields the curves on which the paths lie (except those on the y-axis), but gives no information about the manner in which the paths are traced out. It is clear that the critical point $(0, 0)$ is a node for (5) and it is stable. Refer to Figure 2 for complete visualization.

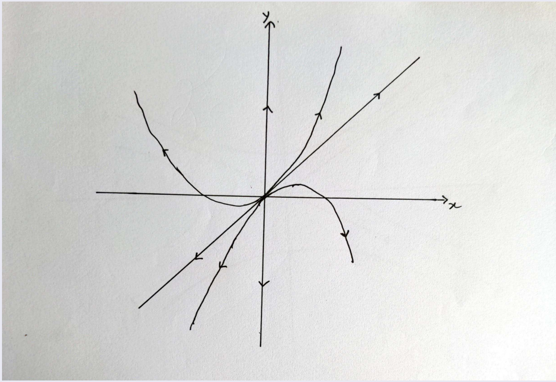
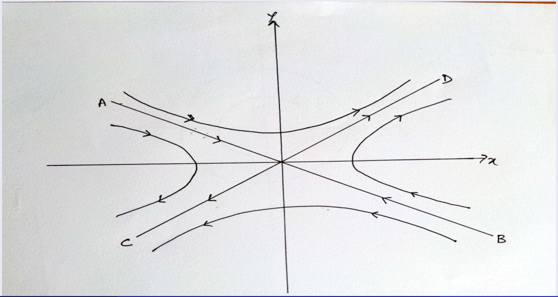


Figure 2

Saddle point

It is a critical point that is approached and entered by two half-line paths AO and BO as $t \to \infty$, and these paths lie on a line AB. It is also approached and entered by two half-line paths CO and DO (lying on line CD) as $t \to \infty$.

Further to it, Between the four half-line paths there are four regions – each containing a family of paths resembling (rectangular) hyperbolas. These paths do not approach O as $t \to \infty$ or as $t \to -\infty$ but are asymptotic to one or another of the half-line paths as $t \to \infty$ and as $t \to -\infty$. (Fig 3)



Example

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = -y.$$

Solution

Here $a_1 = 1, a_2 = 0, b_1 = 0, b_2 = -1$. The auxiliary equation for the system is

$$m^2 - 1 = 0,$$

giving us $m = 1, -1$.

Hence the general solution can be found as

$$\left.\begin{aligned} x &= c_1 e^{t}, \\ y &= c_2 e^{-t}. \end{aligned}\right\} \tag{8}$$

When $c_1 = 0$, then $x = 0, y = c_2 e^{-t}$. In this case, the path is the positive y-axis when $c_2 > 0$ and the negative y-axis when $c_2 < 0$. Each path approaches and enters the origin as $t \to \infty$.

When $c_2 = 0$, then $x = c_1 e^{t}, y = 0$. In this case, the path is the positive x-axis when $c_1 > 0$ and the negative x-axis when $c_1 < 0$. Each path approaches and enters the origin as $t \to -\infty$.

When both c_1 and c_2 are not zero, the paths lie on the (rectangular) hyperbolas $xy = c_1 c_2$, which will never go through the origin.

Clearly $(0, 0)$ is a saddle point which is unstable.

Center

A center, which is also sometimes called a vortex, is a critical point that is surrounded by a family of closed curves. It is not approached by any path as $t \to \infty$ or as $t \to -\infty$.

Example

The system

$$\frac{dx}{dt} = -y, \ \frac{dy}{dt} = x \tag{9}$$

has the origin as its only critical point.

Here $a_1 = 0, a_2 = 1, b_1 = -1, b_2 = 0$. Hence the auxiliary equation $m^2 + 1 = 0$ gives us the roots as $m = \pm \mathrm{i}$.

Therefore, the general solution of (9) can be written as

$$x = -c_1 \sin t + c_2 \cos t, \ \ y = c_1 \cos t + c_2 \sin t. \tag{10}$$

The solution of (10) satisfying the conditions $x(0) = 1$ and $y(0) = 0$ is clearly

$$x = \cos t, \ y = \sin t, \tag{11}$$

and the solutions determined by $x(0) = 0$ and $y(0) = -1$ is

$$x = \sin t = \cos(t - \pi/2), \ y = -\cos t = \sin(t - \pi/2). \tag{12}$$

These two different solutions (11) and (12) define the same path C (Figure 4) which is evidently the circle $x^2 + y^2 = 1$. Both (11) and (12) show that the path is traced out in the anti-clockwise direction.

If we eliminate t between the equations of the system, we get

$$\frac{dy}{dx} = -\frac{x}{y}$$

whose general solution $x^2 + y^2 = c^2$ yields all the paths (but without their directions). Obviously the critical point $(0, 0)$ of the given system (9) is a center.

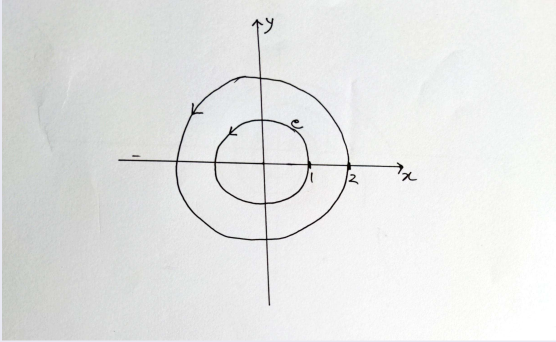


Figure 4

Spiral

A spiral or a focus is a critical point that is approached in a spiral-like manner by a family of paths that wind around it an infinite number of times as $t \to \infty$ or as $t \to -\infty$. Note that although the paths approach the origin O, they actually do not enter it.
That is, a point P moving along such a path approaches O as $t \to \infty$ (or $t \to -\infty$) but the line OP does not approach any definite direction.

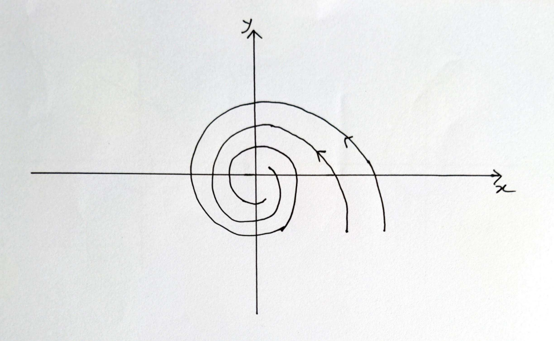


Figure 5

Example

If a is an arbitrary constant, then the system

$$\left.\begin{aligned} \tfrac{dx}{dt} &= ax - y, \\ \tfrac{dy}{dt} &= x + ay \end{aligned}\right\} \tag{13}$$

has the origin as its only critical point.

Solution

The differential equation of the paths

$$\frac{dy}{dx} = \frac{x + ay}{ax - y} \tag{14}$$

can be easily solved by putting $x = r\cos\theta,\ y = r\sin\theta$ from which we know $r^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$.

Therefore, we have

$$r\frac{dr}{dx} = x + y\frac{dy}{dx} \quad \text{and} \quad r^2\frac{d\theta}{dx} = x\frac{dy}{dx} - y.$$

With this result, (14) gives $\dfrac{dr}{d\theta} = ar$, which gives us

$$r = c\, e^{a\theta} \tag{15}$$

The two possible spiral configurations are for $a > 0$ and $a < 0$.

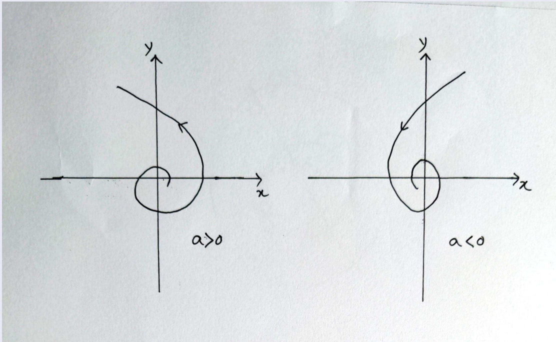


Figure 6

The directions in which these paths are traversed can be seen from the fact that $\frac{dx}{dt} = -ay$ when $x = 0$. If $a = 0$, then $r = c$ which is the polar equation of the family $x^2 + y^2 = c^2$ of all circles centered on origin.

In other words,

this example can be considered as the generalization of the previous example and since the center shown in Figure 4 stands on the borderline between the spirals of Figure 6, a critical point that is a center is often called a borderline case.