MA 542 Differential Equations
Lecture 2
(January 7, 2022)

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a differential equation.

When a differential equation contains only ordinary derivatives, it is called an ordinary differential equation while a differential equation is called a partial differential equation if it contains partial derivatives.

Many of the general laws of nature – in physics, chemistry, biology and astronomy – find their most natural expression in differential equations.

We know that if $y = f(x)$ or $y = f(t)$ is a function, then its derivative $\frac{dy}{dx}$ or $\frac{dy}{dt}$ can be interpreted as the rate of change of y with respect to x or t. In many natural processes, the variables involved and their rates of changes are connected to one another by means of the basic scientific principles that govern the process.
When this connection is expressed in mathematical symbols, the result is quite often a differential equation.

Recall

Newton's second law of motion leads to the differential equations

$$\frac{d^2y}{dt^2} = g, \tag{1}$$

and

$$m\frac{d^2y}{dt^2} = mg - k\frac{dy}{dt}. \tag{2}$$

Equations (1) and (2) are the differential equations that express the essential attributes of the physical processes under consideration.

Recall

Newton's law of cooling which leads to the differential equation

$$\frac{dT}{dt} = -k(T - S). \tag{3}$$

The solution to this equation will then be a function that tracks the complete record of the temperature over time.

Order and degree of a differential equation

The order of a differential equation is the order of the highest derivative present.

The degree of a differential equation is the degree (power) of the highest derivative present.

The most general ODE of n-th order has the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0, \quad (4)$$

where F is some arbitrary function at this moment.

In simple notations for derivatives, (4) can be written as

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \quad (5)$$

A general linear n-th order ODE can be written as

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0, \quad (6)$$

so that no term contains a product of y and any of its derivatives.

The most general first-order ODE has the form

$$F(x, y, y') = 0. \tag{7}$$

Generally speaking, it is very difficult to solve first-order differential equations. Even a simple-looking equation

$$\frac{dy}{dx} = f(x, y) \tag{8}$$

cannot be solved in general, in the sense that no formulas exist for obtaining its solution in all cases.

We are interested mainly in finding solutions of the differential equations of certain types. Our main purpose is to acquire technical facility, we shall completely disregard certain properties like continuity, differentiability etc.

The simplest first-order ODE can be $\dfrac{dy}{dx} = 0$,
which indicates that we are looking for such a y which has a vanishing slope. This happens for $y =$ constant, i.e., a family of straight lines parallel to the x-axis.

Similarly a constant slope condition is $\dfrac{dy}{dx} = c$, from which we can obtain y by

$$y = c \int dx + d,$$

which is a straight line with a slope c and y-intercept d.

As a next step, consider

$$\frac{dy}{dx} = f(x), \tag{9}$$

which can be solved by writing

$$y = \int f(x)dx + c.$$

However, in general

it may not be that easy to evaluate $\int f(x)dx$ since $f(x)$ may not be in an easy form.

The simplest of first-order equations is that in which the variables x and y are separable:

$$\frac{dy}{dx} = f(x, y) \equiv g(x)h(y). \tag{10}$$

For equation (10),

we only have to write this in separated form $dy/h(y) = g(x)dx$ and integrate:

$$\int \frac{dy}{h(y)} = \int g(x)dx + c.$$

Example:

Solve by separation of variables: $\dfrac{dy}{dx} = \dfrac{y}{x-1}$.

Solution:

Separating the variables,

$$\int \frac{dy}{y} = \int \frac{dx}{x-1} + c$$
$$\Rightarrow \ln y = \ln(x-1) + \ln C$$
$$\Rightarrow y = C(x-1),$$

which gives a family of straight lines since C is arbitrary.

Introducing a condition $y(2) = 1$, i.e.,

if the line passes through the point (2,1), then we get $C = 1$ and hence the solution satisfying $y(2) = 1$ is $y = x - 1$.

It is clear that different conditions $y(a) = b$ will give us different straight lines.

At the next level of complexity is the homogeneous equation:

A function $f(x, y)$ is called homogeneous of degree n if

$$f(tx, ty) = t^n f(x, y)$$

for all suitably restricted x, y and t.

Thus $x^2 + y^2$, $\sqrt{x^2 + y^2}$ and $\sin(x/y)$ are homogeneous of degrees $2, 1$ and 0, respectively.

The differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be *homogeneous* if M and N are homogeneous functions of the same degree.

This equation then can be written in the form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \equiv f(x,y)$$

Solution Procedure:

Let $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. This means substitution of the dependent variable y by v.Then
the given equation becomes a differential equation in v. Solving for v and then replacing $v = y/x$ gives the solution for y.

Example:

Verify that the following equation is homogeneous and solve it:

$$(x^2 - 2y^2)\, dx + xy\, dy = 0.$$

Solution: The differential equation can be written as

$$\frac{dy}{dx} = 2\frac{y}{x} - \frac{x}{y}.$$

Using $y = vx$:

$$\begin{aligned} & v + x\frac{dv}{dx} = 2v - \frac{1}{v} \\ \Rightarrow\ & x\frac{dv}{dx} = v - \frac{1}{v} \\ \Rightarrow\ & \frac{v\ dv}{v^2 - 1} = \frac{dx}{x} \\ \Rightarrow\ & \frac{1}{2}\ln(v^2 - 1) = \ln x + \ln c \\ \Rightarrow\ & \sqrt{v^2 - 1} = cx. \end{aligned}$$

Putting back $v = y/x$, the solution is obtained as

$$\sqrt{y^2 - x^2} = cx^2.$$

Exact equations

If we begin with a family of curves $f(x,y) = c$, then its differential equation can be written in the form $df = 0$ or

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0.$$

For example, the family $x^2y^3 = c$ has $2xy^3\, dx + 3x^2y^2\, dy = 0$ as its differential equation.

Now let's turn around the situation and begin with the differential equation

$$M(x,y)\, dx + N(x,y)\, dy = 0. \tag{11}$$

If there happens to be a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N, \tag{12}$$

then (11) can be written in the form

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0 \quad \text{or} \quad df = 0$$

Its general solution is

$$f(x, y) = c.$$

In this case the expression $M\,dx + N(x, y)\,dy$ is said to be an *exact differential*, and (11) is called an *exact differential equation*.

For simple differential equations, it is possible to determine the exactness and find the function by inspection. Consider the following two differential equations:

$$ydx + xdy = 0, \quad \frac{1}{y}\,dx - \frac{x}{y^2}\,dy = 0.$$

The left sides of these equations can easily be recognized as the differentials of xy and x/y, respectively. Hence the general solutions of these equations can be written as $xy = c$ and $x/y = c$.

But in general, the exactness of all equations cannot be ascertained by mere inspection and we have to do some tests of exactness and also find a method for finding the function f.

Suppose equation (11) is exact so that there exists a function f satisfying equations (12). We know that the mixed second order partial derivatives of f can be considered to be equal:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}. \tag{13}$$

This yields

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{14}$$

So equation (14) is a necessary condition for the exactness of (11). Let's take an example to see how we solve an exact equation.

Example:

$e^y dx + (xe^y + 2y)dy = 0.$

Solution:

Here we have $M = e^y$ and $N = xe^y + 2y$. So, $\dfrac{\partial M}{\partial y} = e^y$, and $\dfrac{\partial N}{\partial x} = e^y$.

Hence, since the condition of exactness is satisfied, the equation is exact and there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = e^y \text{ and } \frac{\partial f}{\partial y} = xe^y + 2y.$$

Integrating the first equation with respect to x gives

$$f = \int e^y \, dx + g(y) = xe^y + g(y),$$

so,

$$\frac{\partial f}{\partial y} = xe^y + g'(y).$$

Since this partial derivative must also equal $xe^y + 2y$, we have $g'(y) = 2y$, so $g(y) = y^2$ and $f = xe^y + y^2$.

Hence

$$xe^y + y^2 = c$$

is the desired solution to the given differential equation.