MA 542 Differential Equations

Lecture 18

(February 11, 2022)

Systems in matrix form

Consider a system of first-order equations in n unknowns $x_k(t)$, $k = 1, 2, \ldots, n$

$$\left.\begin{array}{l} x_1'(t) = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}x_n + b_1(t) \\ x_2'(t) = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}x_n + b_2(t) \\ \quad \ldots \quad \ldots \quad \ldots \\ \quad \ldots \quad \ldots \quad \ldots \\ x_n'(t) = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}x_n + b_n(t) \end{array}\right\}, t \in I, \tag{1}$$

where all the functions $a_{ij}, b_i, i, j = 1, 2, \ldots, n$ are given.

Define a matrix $A(t)$ by

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}. \tag{2}$$

Define the column vectors $\bar{b}(t)$ and $\bar{x}(t)$, respectively, by

$$\bar{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \quad \bar{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}. \tag{3}$$

Then the system (1) reduces to

$$\bar{x}' = A(t)\bar{x} + \bar{b}(t), \; t \in I. \tag{4}$$

Equation (4) is a vector matrix representation of a linear non-homogeneous system (1).

If $\bar{b}(t) = 0 \; \forall t \in I$, then (4) reduces to a homogeneous system

$$\bar{x}' = A(t)\bar{x}, \; t \in I. \tag{5}$$

Example

Consider the system of equations

$$x_1' = 5x_1 - 2x_2,$$
$$x_2' = 2x_1 + x_2.$$

This system of two equations can be represented in the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Theorem

Let $A(t)$ be an $n \times n$ matrix that is continuous on a closed and bounded interval I. Then there exists a solution to the initial value problem

$$\bar{x}\,' = A(t)\bar{x}, \quad \bar{x}(t_0) = \bar{x}_0;\ t, t_0 \in I \tag{6}$$

and in addition, the solution is unique.

Many times it is convenient to construct a matrix with solutions of (5) as columns. In other words, consider a set of n solutions of the system (5) and define a matrix ϕ whose columns are these n solutions.

This solution is called a solution matrix since it satisfies the homogeneous matrix differential equation

$$\phi' = A(t)\phi, \ t \in I. \tag{7}$$

If the columns are linearly independent, the matrix ϕ thus obtained is called a fundamental matrix for the system (5). We associate with (5), a homogeneous matrix differential equation

$$X' = \mathcal{A}(t)X, \ t \in I. \tag{8}$$

Theorem

A solution matrix ϕ of (8) on I is a fundamental matrix of (4) on I if and only if $\det\ \phi(t) \neq 0$ for $t \in I$.

Linear systems with constant coefficients

Assuming that A is a constant matrix, we first find the characteristic values of the matrix A. If the characteristic values of the matrix A are known then, in general, a solution can be obtained in an explicit form.

Theorem

Consider a linear homogeneous system with constant matrix:

$$\bar{x}^{\prime} = A(t)\bar{x}, \ t \in I. \tag{9}$$

The general solution of (9) is given by $\bar{x}(t) = \alpha e^{\lambda t}\bar{c}$, where λ is a scalar to be obtained, α an arbitrary constant and $\bar{c}$ is a column matrix.

Procedure

Choose a solution of (9) in the form

$$\bar{x}(t) = e^{\lambda t}\bar{c} \tag{10}$$

where $\bar{c}$ is a constant column vector and λ is a scalar.

Substituting (10) in (9), we get

$$(\lambda I - A)\bar{c} = 0, \tag{11}$$

where I is the identity matrix. The system of linear algebraic homogeneous equations (11) can now be solved for $\bar{c}$.

System (11) has a non-trivial solution $\bar{c}$ iff λ satisfies the condition $\det(\lambda I - A) = 0$. Suppose that $P(\lambda) = \det(\lambda I - A)$.

It is clear that $P(\lambda)$ is a polynomial of degree n. This polynomial $P(\lambda)$ is called the characteristic polynomial of the matrix A and the equation

$$P(\lambda) = 0 \tag{12}$$

is called the characteristic equation for A.

Since (12) is an algebraic equation, it admits n roots which may be distinct, repeated or complex. The roots of (12) are called eigenvalues or characteristic values of A.

If λ_1 is an eigenvalue of A and $\bar{c}_1$ is the non-trivial solution of (11) corresponding to this eigenvalue, then $\bar{c}_1$ is called an *eigenvector* of A corresponding to the eigenvalue λ_1.

Thus, if $\{x_k(t)\},\ k = 1, 2, \ldots, n$ is a set of n linearly independent vector functions corresponding to the eigenvalues λ_k are solutions of (9), then by the principle of superposition, the general solution of the linear system can be written as

$$\bar{x}(t) = \sum_{k=1}^{n} \alpha_k e^{\lambda_k t}\ \bar{c}_k. \tag{13}$$

Our above discussion is based on the assumption that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct.

Example

Consider the following system of three first-order equations:

$$\bar{x}\,' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \bar{x}.$$

Solution

The characteristic equation is given by $\det(\lambda I - A) = 0$.
i.e.,

$$\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -6 & 11 & \lambda - 6 \end{bmatrix} = 0$$

which gives us

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

We obtain the eigenvalues as

$$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$$

The corresponding eigenvectors $\bar{c}_k$, $k = 1, 2, 3$, can be evaluated, respectively, as

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}.$$

Thus, the general solution of the system is

$$\bar{x}(t) = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + \alpha_2 \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} e^{2t} + \alpha_3 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3t},$$

where α_1, α_2 and α_3 are arbitrary constants.

Example

Consider the following system of three first-order equations:

$$\bar{x}\,' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \bar{x}.$$

Solution

The characteristic equation is given by $\det(\lambda I - A) = 0$. This gives us

$$(\lambda - 1)^3 + (\lambda - 1) = 0,$$

i.e., we get the eigenvalues as $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$.

The eigenvector corresponding to $\lambda_1 = 0$ is given by

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0,$$

where $u_i, \ i = 1, 2, 3$ are components $\bar{c}_1$ and we get $u_1 + u_3 = 0, \ u_2 = 0$.

Therefore, $\bar{c}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda_1 = 0$.

The eigenvector corresponding to $\lambda_2 = 1$ is given by

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0,$$

where $v_i, \ i = 1, 2, 3$ are components $\bar{c}_2$ and we get $v_1 = 0, \ v_3 = 0, \ v_2 \neq 0$.

Therefore, $\bar{c}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the eigenvector corresponding to $\lambda_2 = 1$.

The eigenvector corresponding to $\lambda_3 = 2$ is given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0,$$

where $w_i,\ i = 1, 2, 3$ are components $\bar{c}_3$ and we get $w_1 - w_3 = 0,\ w_2 = 0$.

Therefore, $\bar{c}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda_3 = 2$.

Hence, the linear independent solutions corresponding to the given problem are

$$x_1(t) = \bar{c}_1,\ x_2(t) = \bar{c}_2 e^t,\ x_3(t) = \bar{c}_3 e^{2t}.$$

Therefore, the general solution is given by

$$\bar{x}(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_3 x_3(t),$$

where α_1, α_2 and α_3 are arbitrary constants.

Equivalently,

the general solution of the system can be written as

$$\bar{x}(t) = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}.$$

Both examples considered here have distinct eigenvalues only. A similar procedure can be adopted even if the eigenvalues are repeated or are complex.