MA 542 Differential Equations

Lecture 17

(February 10, 2022)

Theorem

Consider the following homogeneous system:

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y, \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y. \end{aligned}\right\} \tag{1}$$

If this system (1) has two solutions

$$\left.\begin{aligned} x &= x_1(t), \\ y &= y_1(t), \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= x_2(t), \\ y &= y_2(t), \end{aligned}\right\} \tag{2}$$

on $[a, b]$, then

$$\left.\begin{aligned} x &= c_1x_1(t) + c_2x_2(t), \\ y &= c_1y_1(t) + c_2y_2(t), \end{aligned}\right\} \tag{3}$$

is also a solution on $[a, b]$ for any constants c_1 and c_2.

Theorem

If the two solutions (2) of the homogeneous system (1) have a Wronskian $W(t)$ that does not vanish on $[a, b]$, then (3) is the general solution of (1) on this interval, where

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}.$$

It can be seen that the vanishing or non-vanishing of $W(t)$ of two solutions does not depend on the choice of t. A formal statement can be made as follows:

Theorem

If $W(t)$ is the Wronskian of the two solutions (2) of the homogeneous system (1), then $W(t)$ is either identically zero or never zero on $[a, b]$.

Theorem

If the two solutions (2) of the homogeneous system (1) are linearly independent on $[a, b]$, then (3) is the general solution of (1) on this interval.

Homogeneous linear systems with constant coefficients

Consider the following simple first-order system:

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1 x + b_1 y, \\ \frac{dy}{dt} &= a_2 x + b_2 y, \end{aligned}\right\} \tag{4}$$

where a_1, a_2, b_1, b_2 are given constants.

Seek solutions of (4) in the form

$$x = Ae^{mt}, \quad y = Be^{mt}, \tag{5}$$

where A and B are constants to be determined. Then (4) will give rise to the following equations:

$$\left.\begin{aligned} (a_1 - m)A + b_1 B = 0, \\ a_2 A + (b_2 - m)B = 0. \end{aligned}\right\} \tag{6}$$

For non-trivial solution for A and B, we must have

$$\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} = 0.$$

This gives

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0, \tag{7}$$

which is known as the auxiliary equation of (4).

Let m_1 and m_2 be the roots of (7). Replacing m in (5) by m_1, we know the resulting equations have nontrivial A_1 and B_1, so that

$$\left.\begin{aligned} x &= A_1 e^{m_1 t}, \\ y &= B_1 e^{m_1 t}, \end{aligned}\right\} \tag{8}$$

is a nontrivial solution of the system (4).

Similarly, proceeding with m_2, we get another nontrivial solution

$$\left.\begin{aligned} x &= A_2 e^{m_2 t}, \\ y &= B_2 e^{m_2 t}. \end{aligned}\right\} \tag{9}$$

We have to find appropriate values of A_1, A_2, B_1 and B_2 so that we can find the solutions (8) and (9). In this regard, we need to utilize (6).

To obtain two linearly independent solutions and hence the general solution, we need to examine three possibilities for m_1 and m_2 with respect to their nature - whether they are real and distinct or real and equal or complex conjugate.

Case I. Distinct real roots.

If m_1 and m_2 are distinct real numbers, then (8) and (9) are linearly independent and

$$\left.\begin{aligned} x &= c_1 A_1 e^{m_1 t} + c_2 A_2 e^{m_2 t}, \\ y &= c_1 B_1 e^{m_1 t} + c_2 B_2 e^{m_2 t}, \end{aligned}\right\} \tag{10}$$

is the general solution of (4).

Example

$$\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = 4x - 2y.$$

Solution:

Here $a_1 = 1, a_2 = 4, b_1 = 1, b_2 = -2$. The auxiliary equation $m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0$ gives

$$m^2 + m - 6 = 0$$

$$\Rightarrow (m + 3)(m - 2) = 0.$$

We get

$m_1 = -3, m_2 = 2$, and for this problem, (6) is

$$\left.\begin{aligned}(1-m)A + B &= 0,\\ 4A + (-2-m)B &= 0.\end{aligned}\right\} \tag{11}$$

With $m_1 = -3$, (11) becomes

$$4A_1 + B_1 = 0,$$

$$4A_1 + B_1 = 0.$$

This gives a simple non-trivial solution as $A_1 = 1, B_1 = -4$ so that we have a non-trivial solution of the given equation as

$$\left.\begin{aligned}x &= A_1 e^{m_1 t} = e^{-3t},\\ y &= B_1 e^{m_1 t} = -4e^{-3t}.\end{aligned}\right\} \tag{12}$$

With $m_2 = 2$, (11) becomes

$$-A_2 + B_2 = 0,$$
$$4A_2 - 4B_2 = 0.$$

This gives a simple non-trivial solution as $A_2 = 1, B_2 = 1$ so that we have another non-trivial solution of the given equation as

$$\left.\begin{aligned} x &= A_2 e^{m_2 t} = e^{2t}, \\ y &= B_2 e^{m_2 t} = e^{2t}. \end{aligned}\right\} \tag{13}$$

Therefore, the general solution of the given equation can be found as

$$\left.\begin{aligned} x &= c_1 e^{-3t} + c_2 e^{2t}, \\ y &= -4c_1 e^{-3t} + c_2 e^{2t}. \end{aligned}\right\} \tag{14}$$

Case II: Complex roots.

If m_1 and m_2 are complex numbers (conjugate pair), they can be written in the form $a \pm \mathrm{i}b, \ a, b \in \mathbb{R}, \ b \neq 0.$

Consider the two linearly independent solutions as

$$\left.\begin{array}{l} x = A_1^* e^{(a+\mathrm{i}b)t}, \\ y = B_1^* e^{(a+\mathrm{i}b)t}, \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} x = A_2^* e^{(a-\mathrm{i}b)t}, \\ y = B_2^* e^{(a-\mathrm{i}b)t}. \end{array}\right\} \tag{15}$$

Writing $A_1^* = A_1 + \mathrm{i}A_2, \ B_1^* = B_1 + \mathrm{i}B_2$, the first pair of the above solutions can be written as

$$x = (A_1 + \mathrm{i}A_2)e^{at}(\cos bt + \mathrm{i}\sin bt),$$
$$y = (B_1 + \mathrm{i}B_2)e^{at}(\cos bt + \mathrm{i}\sin bt).$$

Equivalently,

$$\left.\begin{array}{l} x = e^{at}[(A_1 \cos bt - A_2 \sin bt) + \mathrm{i}(A_1 \sin bt + A_2 \cos bt)], \\ y = e^{at}[(B_1 \cos bt - B_2 \sin bt) + \mathrm{i}(B_1 \sin bt + B_2 \cos bt)]. \end{array}\right\} \tag{16}$$

(16) gives the following two real-valued solutions:

$$\left.\begin{array}{l} x = e^{at}(A_1 \cos bt - A_2 \sin bt), \\ y = e^{at}(B_1 \cos bt - B_2 \sin bt). \end{array}\right\} \quad (17)$$

and

$$\left.\begin{array}{l} x = e^{at}(A_1 \sin bt + A_2 \cos bt), \\ y = e^{at}(B_1 \sin bt + B_2 \cos bt). \end{array}\right\} \quad (18)$$

This is due to the following:

If a pair of complex-valued functions is a solution of (4) in which the coefficients are real, then their two real parts and two imaginary parts are real-valued functions which represent two solutions.

These solutions (17) and (18) are linearly independent and hence the general solution is

$$\left.\begin{array}{l} x = e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)], \\ y = e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)]. \end{array}\right\} \quad (19)$$

Since we have already found the general solution, it is not necessary to consider the second of the two solutions (15).

Example

$$\frac{dx}{dt} = 4x - 2y, \quad \frac{dy}{dt} = 5x + 2y.$$

Solution:

Here $a_1 = 4, a_2 = 5, b_1 = -2, b_2 = 2$. The auxiliary equation $m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0$ gives

$$m^2 - 6m + 18 = 0.$$

We get the roots as

$$m = 3 \pm 3\mathrm{i}.$$

Hence the general solution can be written as

$$\left.\begin{aligned} x &= e^{3t}[c_1(A_1\cos 3t - A_2\sin 3t) + c_2(A_1\sin 3t + A_2\cos 3t)], \\ y &= e^{3t}[c_1(B_1\cos 3t - B_2\sin 3t) + c_2(B_1\sin 3t + B_2\cos 3t)]. \end{aligned}\right\}$$

A_1, A_2, B_1 and B_2 can be obtained as $A_1 = 2, A_2 = 4, B_1 = 1$ and $B_2 = 10$ by using (6).

Hence the general solution ultimately takes the form

$$\left.\begin{aligned} x &= e^{3t}[c_1(2\cos 3t - 4\sin 3t) + c_2(2\sin 3t + 4\cos 3t)], \\ y &= e^{3t}[c_1(\cos 3t - 10\sin 3t) + c_2(\sin 3t + 10\cos 3t)]. \end{aligned}\right\}$$

Case III: Equal real roots.

When m_1 and m_2 have the same value m, then (8) and (9) are not linearly independent and we essentially have only one solution

$$\left.\begin{aligned} x &= Ae^{mt}, \\ y &= Be^{mt}. \end{aligned}\right\} \tag{20}$$

We must look for a second solution of the form

$$\left.\begin{aligned} x &= (A_1 + A_2 t)e^{mt}, \\ y &= (B_1 + B_2 t)e^{mt}. \end{aligned}\right\} \tag{21}$$

This allows us to write the general solution as

$$\left.\begin{aligned} x &= c_1 Ae^{mt} + c_2(A_1 + A_2 t)e^{mt}, \\ y &= c_1 Be^{mt} + c_2(B_1 + B_2 t)e^{mt}. \end{aligned}\right\} \tag{22}$$

The constants A and B can be found by using (6).

The constants A_1, A_2, B_1 and B_2 can be found by substituting (21) into the system (4).

Example

$$\frac{dx}{dt} = 3x - 4y, \quad \frac{dy}{dt} = x - y. \tag{23}$$

Solution:

We have, $a_1 = 3, a_2 = 1, b_1 = -4, b_2 = -1$ and

$$\left.\begin{array}{r} (3-m)A - 4B = 0, \\ A + (-1-m)B = 0. \end{array}\right\} \tag{24}$$

The auxiliary equation turns out to be

$$m^2 - 2m + 1 = 0,$$

which gives the root as $m = 1$

With $m = 1$, (24) becomes

$$2A - 4B = 0,$$
$$A - 2B = 0.$$

Finding A and B as $A = 2$ and $B = 1$, we have a non-trivial solution as

$$\left.\begin{aligned} x &= 2e^t, \\ y &= e^t. \end{aligned}\right\} \qquad (25)$$

Now we seek a second linearly independent solution of the form

$$\left.\begin{aligned} x &= (A_1 + A_2 t)e^t, \\ y &= (B_1 + B_2 t)e^t. \end{aligned}\right\} \qquad (26)$$

When (26) is substituted in (23), we have

$$(A_1 + A_2 t + A_2)e^t = 3(A_1 + A_2 t)e^t - 4(B_1 + B_2 t)e^t,$$
$$(B_1 + B_2 t + B_2)e^t = (A_1 + A_2 t)e^t - (B_1 + B_2 t)e^t.$$

This system in unknowns A_1, A_2, B_1 and B_2 reduces to

$$(2A_2 - 4B_2)t + (2A_1 - A_2 - 4B_1) = 0,$$
$$(A_2 - 2B_2)t + (A_1 - 2B_1 - B_2) = 0.$$

Since the above are identities in the variable t, we must have

$$2A_2 - 4B_2 = 0, \qquad 2A_1 - A_2 - 4B_1 = 0,$$
$$A_2 - 2B_2) = 0, \qquad A_1 - 2B_1 - B_2 = 0.$$

System of linear equations

The two equations on the left have simple non-trivial solutions as $A_2 = 2, B_2 = 1$. Taking these values, the equations on the right become

$$2A_1 - 4B_1 = 2, \quad A_1 - 2B_1 = 1.$$

We may take $A_1 = 1, B_1 = 0$. Insert these four values into (26) to obtain the second solution as

$$\left.\begin{aligned} x &= (1+2t)e^t, \\ y &= te^t. \end{aligned}\right\} \tag{27}$$

It is obvious that (25) and (27) are linearly independent and hence the general solution of (23):

$$\left.\begin{aligned} x &= 2c_1e^t + c_2(1+2t)e^t, \\ y &= c_1e^t + c_2te^t. \end{aligned}\right\} \tag{28}$$

Thus we have observed that different types of solutions emerge depending upon the nature of the roots of the auxiliary equation corresponding to a system of two linear homogeneous first-order equations. The trend is similar to what happens when we consider a second-order linear homogeneous ODE with constant coefficients.

A similar procedure will work even if we have more than two first-order equations.