MA 542 Differential Equations

Lecture 16

(February 7, 2022)

Recall:

Picard's Theorem: Let $f(x, y)$ and $\partial f / \partial y$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axes. If (x_0, y_0) is any interior point of R, then there exists a number $h > 0$ with the property that the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

has one and only one solution $y = y(x)$ on the interval $|x - x_0| \leq h$.

We know that every solution of (1) is also a continuous solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]dt, \tag{2}$$

and conversely.

Remark 1.

Picard's theorem can be strengthened in various ways by weakening its hypotheses. For instance, our assumption that $\partial f/\partial y$ is continuous on R is stronger than the proof requires, and is used only to obtain the inequality

$$|f(x,y_1)-f(x,y_2)| = \left|\frac{\partial}{\partial y} f(x,s)\right| |y_1-y_2|. \tag{3}$$

Remark 1. (Cont.)

We, therefore, introduce this inequality into the theorem as an assumption that replaces the assumption about $\partial f/\partial y$. Consequently, we arrive at a stronger form of the theorem since there are many functions that lack a continuous partial derivative but satisfy (3) for some constant K.

Remark 1. (Cont.)

This inequality, which says that the difference quotient

$$\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2}$$

is bounded on R, is called a Lipschitz condition in the dependent variable y.

Remark 2.

If we drop the Lipschitz condition, and assume only that $f(x, y)$ is continuous on R, it is still possible to prove that the initial value problem $y'(x) = f(x, y), \;\; y(x_0) = y_0$ has a solution. This modified result is known as Peano's theorem.

Remark 2. (Cont.)

The only known proofs depend on more sophisticated arguments than those we have used in our present analysis. Furthermore, the solution whose existence Peano's theorem guarantees is not necessarily unique.

Remark 2. (Cont.)

As an example, consider the problem

$$y' = 3y^{2/3}, \; y(0) = 0 \tag{4}$$

and let R be the rectangle $|x| \leq 1, |y| \leq 1$. Here $f(x, y) = 3y^{2/3}$ is plainly continuous on R. Further, $y_1(x) = x^3$ and $y_2(x) = 0$ are two different solutions valid for all x, so (4) certainly has a solution which is not unique.

Remark 2. (Cont.)

The explanation for this nonuniqueness lies in the fact that $f(x, y)$ does not satisfy a Lipschitz condition on the rectangle R since the difference quotient

$$\frac{f(0, y) - f(0, 0)}{y - 0} = \frac{3y^{2/3}}{y} = \frac{3}{y^{1/3}}$$

is unbounded in every neighbourhood of the origin.

Remark 3.

Picard's theorem is called a local existence and uniqueness theorem because it guarantees the existence of a unique solution only on some interval $|x - x_0| \leq h$ where h may be very small. However, there are several important cases in which this restriction can be removed.

Remark 3. (Cont.)

Let us consider the first-order linear equation

$$y' + P(x)y = Q(x),$$

where $P(x)$ and $Q(x)$ are defined and continuous on an interval $a \leq x \leq b$.

Remark 3. (Cont.)

Here we have

$$f(x, y) = -P(x)y + Q(x),$$

and if $K = \max|P(x)|$ for $a \leq x \leq b$, it is clear that

$$|f(x, y_1) - f(x, y_2)| = |-P(x)(y_1 - y_2)| \leq K|y_1 - y_2|.$$

Remark 3. (Cont.)

The function $f(x, y)$ is, therefore, continuous and satisfies a Lipschitz condition on the infinite vertical strip defined by $a \leq x \leq b$ and $-\infty < y < \infty$.

Remark 3. (Cont.)

Under these circumstances, the initial value problem

$$y' + P(x)y = Q(x), \quad y(x_0) = y_0$$

has a unique solution on the entire interval $a \leq x \leq b$. Furthermore, the point (x_0, y_0) can be any point of the strip, interior or not.

One of the fundamental concepts of analysis is that of a system of n simultaneous first-order equations. If $y_1(x), y_2(x), \ldots, y_n(x)$ are n unknown functions of a single independent variable x, then the most general system of interest is one in which their derivatives $y_1', y_2', \ldots, y_n'$ are explicitly given as functions of x and $y_1(x), y_2(x), \ldots, y_n(x)$.

We can consider the following:

$$\left.\begin{aligned} y_1'(x) &= f_1(x, y_1, y_2, \ldots, y_n), \\ y_2'(x) &= f_2(x, y_1, y_2, \ldots, y_n), \\ \ldots \quad & \ldots \\ y_n'(x) &= f_n(x, y_1, y_2, \ldots, y_n). \end{aligned}\right\} \tag{5}$$

Systems of first-order differential equations arise quite naturally in many scientific problems.

An important mathematical reason for studying systems is that the single n-th order equation

$$y^{(n)}(x) = f(x, y, y', y'', \ldots, y^{(n-1)}) \tag{6}$$

can always be regarded as a special case of (5).

To observe this, we put

$$y_1 = y,\ y_2 = y', \ldots,\ y_n = y^{(n-1)} \tag{7}$$

and observe that (6) is equivalent to the system

$$\left.\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ \cdots&\cdots, \\ y_n' &= f(x, y_1, y_2, \ldots, y_n). \end{aligned}\right\} \tag{8}$$

(8) is clearly a special case of (5). The statement that (6) and (8) are equivalent means
(i) if $y(x)$ is a solution of equation (6), then the functions $y_1(x), y_2(x), \ldots, y_n(x)$ defined by (7) satisfy (8);
(ii) and conversely, if $y_1(x), y_2(x), \ldots, y_n(x)$ defined by (7) satisfy (8), then $y(x) = y_1(x)$ is a solution of (6).

This reduction of an n-th order equation to a system of n first-order equations has several advantages. However, here we restrict our attention to systems of only first-order equations in two unknown variables.

That is, we consider a system of the following form:

$$\left.\begin{aligned} \frac{dx}{dt} &= F(t,x,y), \\ \frac{dy}{dt} &= G(t,x,y). \end{aligned}\right\} \tag{9}$$

The system

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y + f_1(t), \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y + f_2(t), \end{aligned}\right\} \tag{10}$$

is called a linear system of two first-order equations since both variables x and y appear linearly. Such equations have a number of applications in mathematical modelling of physical problems.

We assume that the functions $a_i(t), b_i(t)$ and $f_i(t), i = 1, 2,$ are continuous on a certain interval $[a, b]$ of the t-axis. If $f_1(t)$ and $f_2(t)$ are both identically zero, then the system (10) is called homogeneous, otherwise non-homogeneous.

A solution of (10) on $[a, b]$ is a pair of functions $x(t)$ and $y(t)$ that satisfy both the equations of (10) throughout this interval. The solution will be written in the form

$$\left.\begin{aligned} x &= x(t), \\ y &= y(t). \end{aligned}\right\}$$

We now state some useful relevant results which will be required to discuss the solutions of such linear systems.