MA 542 Differential Equations

Lecture 15

(February 4, 2022)

Picard's Theorem

Let $f(x,y)$ and $\partial f/\partial y$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axes. If (x_0, y_0) is any interior point of R, then there exists a number $h > 0$ with the property that the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0 \tag{1}$$

has one and only one solution $y = y(x)$ on the interval $|x - x_0| \leq h$.

Proof: The argument is fairly long and hence we will carry out the proof in a number of steps.

First, we know that every solution of (1) is also a continuous solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]dt, \tag{2}$$

and conversely.

This enables us to conclude that (1) has a unique solution on an interval $|x - x_0| \leq h$ if and only if (2) has a unique continuous solution on the same interval.

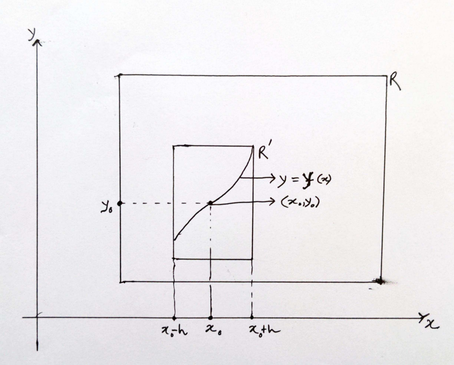

Figure 1 : Geometrical view of the problem: The rectangle R' contained in R

Previously, we have seen that the sequence of functions $\{y_n(x)\}$ defined by

$$\left.\begin{array}{rcl} y_0(x) & = & y_0, \\ y_1(x) & = & y_0 + \int_{x_0}^{x} f(t, y_0)\, dt, \\ y_2(x) & = & y_0 + \int_{x_0}^{x} f[t, y_1(t)]\, dt, \\ & & \dots \\ y_n(x) & = & y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\, dt, \\ & & \dots \end{array}\right\} \tag{3}$$

converges to a solution of (2).

We next observe that $y_n(x)$ is the n-th partial sum of the series of functions

$$\begin{aligned} y_0(x) + \sum_{n=1}^{\infty}[y_n(x) - y_{n-1}(x)] &= y_0(x) + [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] \\ &\quad + \cdots + [y_n(x) - y_{n-1}(x)] + \cdots . \end{aligned} \tag{4}$$

So, the convergence of the sequence (3) is equivalent to the convergence of this series in (4). In order to complete the proof, we produce a number $h > 0$ that defines the interval $|x - x_0| \le h$.

Then we show that, on this interval, the following statements are true:
(i) the series (4) converges to a function $y(x)$,
(ii) $y(x)$ is a continuous solution of (2), and
(iii) $y(x)$ is the only continuous solution of (2).

The hypotheses of the theorem are used to produce the positive number h:

We have assumed that $f(x, y)$ and $\partial f/\partial y$ are continuous functions on the rectangle R. But since R is closed, in the sense that it includes its boundary, and bounded, so each of these functions is necessarily bounded on R.

This means that there exist constants M and K such that

$$|f(x, y)| \leq M, \tag{5}$$

$$\left|\frac{\partial}{\partial y} f(x, y)\right| \leq K, \tag{6}$$

for all points (x, y) in R.

We next observe that if (x, y_1) and (x, y_2) are distinct points in R with the same x coordinate, then the mean value theorem guarantees that

$$|f(x, y_1) - f(x, y_2)| = \left|\frac{\partial}{\partial y} f(x, s)\right| |y_1 - y_2| \tag{7}$$

for some number s between y_1 and y_2.

It is clear from (6) and (7) that

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2| \tag{8}$$

for any points (x, y_1) and (x, y_2) in R (distinct or not) that lie on the same vertical line.

We now choose h to be any positive number such that

$$Kh < 1 \tag{9}$$

and the rectangle R' defined by the inequalities $|x - x_0| \leq h$ and $|y - y_0| \leq Mh$ is contained in R.

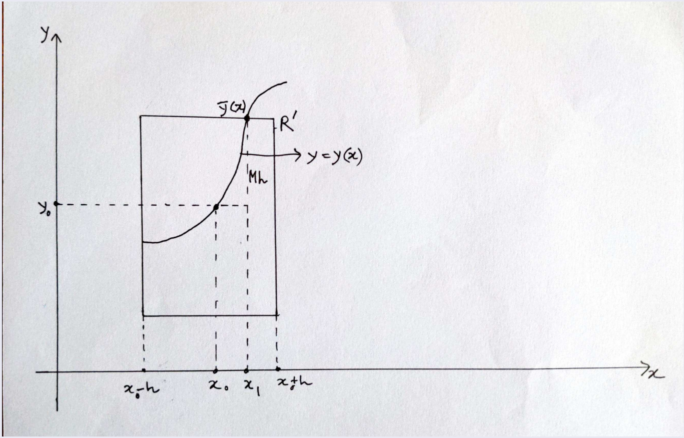

Figure 2 : Definition of Mh

Since (x_0, y_0) is an interior point of R, there is no difficulty in seeing that such an h exists. The reasons for these apparently unreasonable requirements will of course emerge as the proof continues.

From now onward, we confine our attention only to the interval $|x - x_0| \leq h$. In order to prove (i), it suffices to show that the series

$$|y_0(x)| + |y_1(x) - y_0(x)| + |y_2(x) - y_1(x)| + \cdots + |y_n(x) - y_{n-1}(x)| + \cdots \tag{10}$$

converges; and to accomplish this, we estimate the terms $|y_n(x) - y_{n-1}(x)|$.

It is first necessary to observe that each of the functions $y_n(x)$ has a graph that lies in R' and hence in R. This is obvious for $y_0(x) = y_0$, so the points $[t, y_0(t)]$ are in R', (5) yields $|f[t, y_0(t)]| \leq M$, and

$$|y_1(x) - y_0| = \left| \int_{x_0}^{x} f[t, y_0(t)]dt \right| \leq Mh,$$

which proves the statement for $y_1(x)$.

It follows in turn from this inequality that the points $[t, y_1(t)]$ are in R', so $|f[t, y_1(t)]| \leq M$ and

$$|y_2(x) - y_0| = \left| \int_{x_0}^{x} f[t, y_1(t)]dt \right| \leq Mh.$$

Similarly,

$$|y_3(x) - y_0| = \left| \int_{x_0}^{x} f[t, y_2(t)]dt \right| \leq Mh,$$

and so on.

Now for the estimates mentioned above, since a continuous function on a closed interval has a maximum, and $y_1(x)$ is continuous, we can define a constant a by $a = \max|y_1(x) - y_0|$ and write

$$|y_1(x) - y_0(x)| \leq a.$$

Next, the points $[t, y_1(t)]$ and $[t, y_0(t)]$ lie in R', so (8) yields

$$|f[t, y_1(t)] - f[t, y_0(t)]| \leq K|y_1(t) - y_0(t)| \leq Ka.$$

We have

$$|y_2(x) - y_1(x)| = \left| \int_{x_0}^{x} (f[t, y_1(t) - f[t, y_0(t)])dt \right| \le Kah = a(Kh).$$

Similarly,

$$|f[t, y_2(t)] - f[t, y_1(t)]| \le K|y_2(t) - y_1(t)| \le K^2 ah.$$

Therefore,

$$|y_3(x) - y_2(x)| = \left| \int_{x_0}^{x} (f[t, y_2(t) - f[t, y_1(t)])dt \right| \le (K^2 ah)h = a(Kh)^2.$$

By continuing in this manner, we find that

$$|y_n(x) - y_{n-1}(x)| \le a(Kh)^{n-1}$$

for every $n = 1, 2, \ldots$. Each term of the series (10) is therefore less than or equal to the corresponding term of the series of constants

$$|y_0| + a + a(Kh) + a(Kh)^2 + \cdots + a(Kh)^{n-1} + \cdots .$$

However, (9) guarantees that this series converges and hence (10) converges by the comparison test, and
(4) converges to a sum denoted by $y(x)$, and $y_n(x) \to y(x)$.

Since the graph of each $y_n(x)$ lies in R', it is evident that the graph of $y(x)$ also has this property. This proves part (i), i.e., the series (4) converges to a function $y(x)$.

Now we come to the proof of (ii).

The above argument shows not only that $y_n(x)$ converges to $y(x)$ in the interval, but also that this convergence is uniform. This means that by choosing n to be sufficiently large, we can make $y_n(x)$ as close as we please to $y(x)$ for all x in the interval.

More precisely, if

$\epsilon > 0$ is given, then there exists a positive number n_0 such that if $n \geq n_0$, we have $|y(x) - y_n(x)| < \epsilon$ for all x in the interval.

Since each $y_n(x)$ is clearly continuous, this uniformity of the convergence implies that the limit function $y(x)$ is also continuous. To prove that $y(x)$ is actually a solution of (2), we must show that

$$y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]dt = 0. \tag{11}$$

But we know that

$$y_n(x) - y_0 - \int_{x_0}^{x} f[t, y_{n-1}(t)]dt = 0 \tag{12}$$

Hence,

$$y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]dt = y(x) - y_n(x) + \int_{x_0}^{x} (f[t, y_{n-1}(t) - f[t, y(t)])dt.$$

Hence, we obtain

$$\left| y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]dt \right| \leq |y(x) - y_n(x)| + \left| \int_{x_0}^{x} (f[t, y_{n-1}(t) - f[t, y(t)])dt \right|.$$

Since the graph of $y(x)$ lies in R' and hence in R, (8) yields

$$\left|y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]dt\right| \leq |y(x) - y_n(x)| + Kh\max|y_{n-1}(x) - y(x)|. \tag{13}$$

More precisely,

the uniformity of the convergence of $y_n(x)$ to $y(x)$ now implies that the right side of (13) can be made as small as we please by taking n large enough. The left side of (13) must therefore be zero, and the proof of (11) is complete. Consequently, part (ii) of the proof is done, i.e., $y(x)$ is a continuous solution of (2).

In other words,

we have shown that (2) has a solution $y(x)$ which in turn implies that the IVP (1) has a solution $y(x)$. Our next task is to establish (iii), i.e.,
$y(x)$ is the only solution for IVP (1).

In order to prove (iii), we assume that $\bar{y}(x)$ is also a continuous solution of (2) on the interval $|x - x_0| \leq h$, and we show that $\bar{y}(x) = y(x)$ for every x in the interval.

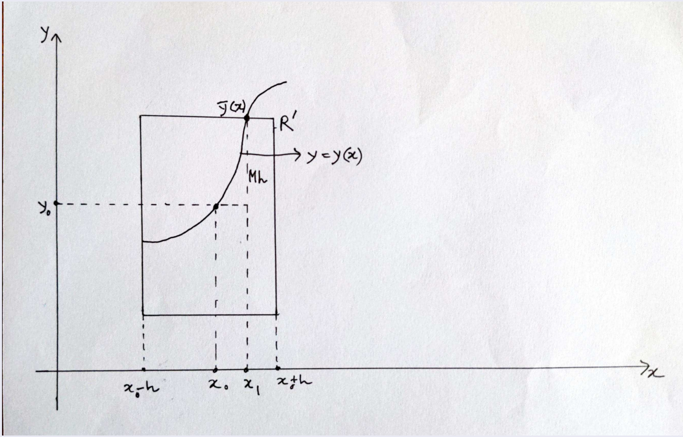

Figure 3 : For showing uniqueness of solution

Existence and uniqueness of solution

For the given argument, it is necessary to know that the graph of $\bar{y}(x)$ lies in R' and hence in R and hence our first step is to establish this fact.

Let us suppose that the graph of $\bar{y}(x)$ leaves R'. Then the properties of this function (continuity and the fact that $\bar{y}(x_0) = y_0$) imply that there exists an x_1 such that $|x_1 - x_0| < h$, $|\bar{y}(x_1) - y_0| = Mh$, and $|\bar{y}(x) - y_0| < Mh$ if $|x - x_0| < |x_1 - x_0|$.

It follows that

$$\frac{|\bar{y}(x) - y_0|}{|x_1 - x_0|} = \frac{Mh}{|x_1 - x_0|} > \frac{Mh}{h} = M.$$

However, by the mean value theorem there exists a number x^* between x_0 and x_1 such that

$$\frac{|\bar{y}(x) - y_0|}{|x_1 - x_0|} = |\bar{y}'(x^*)| = |f[x^*, \bar{y}'(x^*)]| \leq M$$

since the point $[x^*, \bar{y}(x^*)]$ lies in R'.

This contradiction shows that no point with the properties of x_1 can exist and hence the graph of $\bar{y}(x)$ lies in R'.

To complete the proof of (iii), we use the fact that $\bar{y}(x)$ and $y(x)$ are both solutions of (2) to write

$$|\bar{y}(x) - y(x)| = \left| \int_{x_0}^{x} \{f[t, \bar{y}(t)] - f[t, y(t)]\} \, dt \right|.$$

Since the graphs of $\bar{y}(x)$ and $y(x)$ both lie in R', (8) yields

$$|f(x, \bar{y}(x)) - f(x, y(x))| \leq Kh \max|\bar{y}(x) - y(x)|$$
$$\Rightarrow |\bar{y}(x) - y(x)| \leq Kh \max|\bar{y}(x) - y(x)|,$$

so that

$$\max|\bar{y}(x) - y(x)| \leq Kh \max|\bar{y}(x) - y(x)|.$$

This implies that $\max|\bar{y}(x) - y(x)| = 0$, for otherwise we would have $1 \leq Kh$ in contradiction to (9). It follows that $\bar{y}(x) = y(x)$ for every x in the interval $|x - x_0| \leq h$, and hence Picard's theorem is fully proved.
That is, we have established that (2) and in turn the IVP (1) has a unique solution.