MA 542 Differential Equations

Lecture 14

(February 3, 2022)

We have learnt procedures of solving differential equations. But we have not yet looked into the qualitative theory of differential equations. Given a problem, we need to check whether there exists a solution for this differential equation and if it exists, whether it is unique.

Well-posed problem

A problem involving a differential equation is called well-posed if the following are satisfied:

1. A solution exists,
2. the solution is unique, and
3. the solution depends continuously on the given input.

Otherwise it is called an ill-posed problem.

Preliminaries

Here we consider a class of functions satisfying the Lipschitz condition which plays an important role in the qualitative theory of differential equations.

Definition

A function $f(x, y)$ defined in a region $D \subset \mathbb{R}^2$ is said to satisfy a *Lipschitz condition* in the dependent variable y with a Lipschitz constant K if the inequality

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2| \tag{1}$$

holds whenever (x, y_1) and (x, y_2) are in D. In such a case, we denote f to be a member of the class Lip (D, K).

As a consequence of definition (1), a function $f(x, y)$ satisfies Lipschitz condition if and only if there exists a constant $K > 0$ such that

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} \leq K, \; y_1 \neq y_2,$$

whenever (x, y_1) and (x, y_2) belong to D.

Theorem

Let $f(x, y)$ be a continuous function defined over a rectangle $R = \{(x, y) : |x - x_0| \leq p, |y - y_0| \leq q\}$. Here p, q are some positive real numbers. Let $\frac{\partial f}{\partial y}$ be defined and continuous on R. Then $f(x, y)$ satisfies Lipschitz condition in R.

Proof:

Since $\frac{\partial f}{\partial y}$ is continuous on R, there exists a positive constant A such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \leq A \tag{2}$$

for all (x, y) in R.

Let (x, y_1) and (x, y_2) be any two points in R. Then by the mean value theorem of differential calculus, there exists a number s which lies between y_1 and y_2 such that

$$f(x, y_1) - f(x, y_2) = \frac{\partial f}{\partial y}(x, s)(y_1 - y_2).$$

Since the point (x, s) lies in R and the inequality (2) holds, it is clear that

$$\left|\frac{\partial f}{\partial y}(x, s)\right| \leq A.$$

Hence, we have

$$|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|,$$

whenever (x, y_1) and (x, y_2) are in R, which completes the proof.

Example

Let $f(x, y) = |y|$ on the unit square R around the origin, namely,

$$R = \{(x, y) : |x| \leq 1, |y| \leq 1\}.$$

The partial derivative of f at $(x, 0)$ fails to exist but f satisfies Lipschitz condition in y on R with Lipschitz constant $K = 1$.

The following example shows that there exist functions which do not satisfy the Lipschitz condition.

Example: Let $f(x, y) = y^{1/2}$ be defined on the rectangle

$$R = \{(x, y) : |x| \leq 2, |y| \leq 2\}.$$

Justification

In this case, f does not satisfy the inequality (1) in R. This is because

$$\frac{f(x, y) - f(x, 0)}{y - 0} = y^{-1/2}$$

is unbounded in R, since it can be made as large as possible by choosing y close to zero.

The Method of Successive Approximations

From the methods of solving differential equations, we have observed that only a few types of differential equations can be solved explicitly in terms of elementary functions, and a few more by the power series method. Still many differential equations fall outside these categories and we do not have any means of solving them. Approximation is the only way out.

Consider the following initial value problem:

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{3}$$

where $f(x, y)$ is an arbitrary function defined and continuous in some neighbourhood of the point (x_0, y_0).

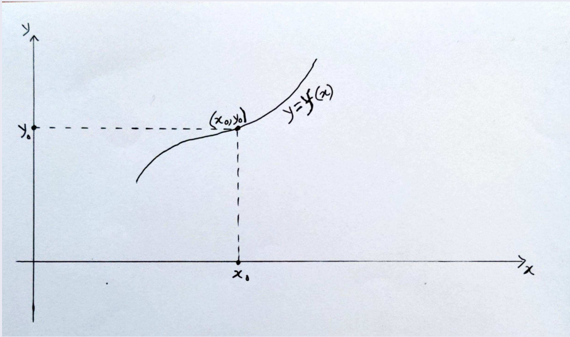

Figure 1 : The curve $y = y(x)$ passing through the point (x_0, y_0)

From the point of view of geometry,

our aim is to devise a method for constructing a function $y = y(x)$ whose graph passes through the point (x_0, y_0) and that satisfies the differential equation $y' = f(x, y)$ in some neighbourhood of x_0.

We replace the initial value problem (3) by the equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]dt. \tag{4}$$

(It is a simple case of integration of the equation in (3) and applying the initial condition.)

Equation (4) is called an integral equation because the unknown function y also occurs under the integral sign.

To see that (3) and (4) are equivalent, suppose that $y(x)$ is a solution of (3). Then $y(x)$ is automatically continuous and the right side of

$$y' = f[x, y(x)]$$

is a continuous function of x; and when we integrate this from x_0 to x and use $y(x_0) = y_0$, the result is (4).

We adopt the process of iteration.

We begin with a rough approximation to a solution and improve it step by step by applying a repeated operation which is supposed to take us closer to the exact solution.

The main advantage that (4) has over (3) is that the integral equation provides a convenient mechanism for carrying out the process.

A rough approximation to a solution is given by the constant function $y_0(x) = y_0$, which is simply a horizontal straight line through the point (x_0, y_0). Insert this expression in the right side of (4) in order to obtain a better approximation $y_1(x)$:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt.$$

The next step is to use $y_1(x)$ to generate another, and perhaps a better approximation $y_2(x)$ in the same way:

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t, y_1(t)]\, dt.$$

After repeating this process for n times, we reach an approximation of the following form:

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\, dt. \tag{5}$$

This procedure is called Picard's method of successive approximations. We will look at the convergence of $y_n(x)$ when $n \to \infty$.

Consider the following simple example:

$$y' = y, \quad y(0) = 1,$$

which has the obvious solution $y(x) = e^x$ and here $f(x, y) = y$.

The equivalent integral equation of above IVP is

$$y(x) = 1 + \int_0^x y(t)\, dt$$

and (5) becomes

$$y_n(x) = 1 + \int_{x_0}^x y_{n-1}(t)\, dt$$

With $y_0(x) = 1$, it is easy to see that

$$y_1(x) = 1 + \int_0^x dt = 1 + x, \quad y_2(x) = 1 + \int_0^x (1+t)dt = 1 + x + \frac{x^2}{2},$$
$$y_3(x) = 1 + \int_0^x (1 + t + t^2/2)dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Continuing this way, in general

$$y_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

We can clearly observe that the successive approximations do really converge to the exact solution ($y_n(x) \to y(x) = e^x$) as the continuation of right side of the above is nothing but the power series expansion of e^x.

Example

Now consider the following first order linear equation

$$y' = x + y, \;\; y(0) = 1,$$

whose exact solution is $y(x) = 2e^x - x - 1$ and here $f(x, y) = x + y$.

The equivalent integral equation is

$$y(x) = 1 + \int_0^x [t + y(t)]dt$$

and (5) gives

$$y_n(x) = 1 + \int_0^x [t + y_{n-1}(t)]dt.$$

With $y_0(x) = 1$, Picard's method yields

$$y_1(x) = 1 + \int_0^x (t+1)dt = 1 + x + \frac{x^2}{2!},$$

$$y_2(x) = 1 + \int_0^x (1 + 2t + t^2/2!)dt = 1 + x + x^2 + \frac{x^3}{3!},$$

$$y_3(x) = 1 + \int_0^x (1 + 2t + t^2 + t^3/3!)dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4!}.$$

Similarly

$$y_4(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3 \cdot 4} + \frac{x^5}{5!}.$$

Continuing this way, in general

$$y_n(x) = 1 + x + 2\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) + \frac{x^{n+1}}{(n+1)!} + \cdots,$$

which evidently converges to (as $n \to \infty$)

$$y(x) = 1 + x + 2(e^x - x - 1) + 0 = 2e^x - x - 1.$$

The real power of Picard's method lies mainly in the theory of differential equations – not in actually finding solutions, but in establishing, under very general conditions, that an initial value problem has a solution and that this solution is unique.

Therefore, our next task is to, given an IVP, find out what are the conditions required to ensure that the IVP has a solution which is unique. This involves rigorous analysis and it is a fairly long process. For convenience, we will deal with only a first-order IVP but the analysis involved will be similar for all order IVPs.