MA 542 Differential Equations

Lecture 13

(February 1, 2022)

Generating function of Legendre polynomial

The function $G(x,t) = \dfrac{1}{\sqrt{1-2xt+t^2}}$ is the generating function with

$$\begin{aligned} \frac{1}{\sqrt{1-2xt+t^2}} &= P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots + P_n(x)t^n + \cdots \\ &= \sum_{n=0}^{\infty} t^n P_n(x), \ |t| < 1. \end{aligned} \tag{1}$$

Many important properties and results of Legendre polynomials can be obtained from the above relation.

Put $x = 1$ in (1):

$$(1-t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(1).$$

Equating the coefficient of t^n from both sides

$$P_n(1) = 1. \tag{2}$$

Similarly, putting $x = -1$ in (1):

$$P_n(-1) = (-1)^n. \tag{3}$$

Now, putting $x = 0$ in (1):

$$(1 + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(0).$$

which gives

$$1 - \frac{1}{2}t^2 + \frac{1 \cdot 3}{2^2 \cdot 2!}t^4 - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}t^6 + \cdots = \sum_{n=0}^{\infty} t^n P_n(0).$$

In above, since all powers of t are even, equating the coefficients of t^{2n-1},

$$P_{2n-1}(0) = 0. \tag{4}$$

The above result is also obvious from the expressions of $P_1(x), P_3(x)$ etc. each of which has all terms having odd powers of x and no constant term.

Next, equating the coefficients of t^{2n},

$$\begin{aligned} P_{2n}(0) &= (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2^n n!} \\ &= (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \text{ (by multiplying numerator and denominator by } 2 \cdot 4 \cdots 2n\text{).} \quad (5) \end{aligned}$$

Look at the expressions of $P_0(x), P_2(x), P_4(x)$ etc. each of which has a constant term in addition to terms containing even powers of x.

Next differentiate (1) with respect to t to get

$$(1 - 2xt + t^2)^{-3/2}(x - t) = \sum_{n=1}^{\infty} n t^{n-1} P_n(x),$$

which can be written as

$$(x - t) \sum_{n=0}^{\infty} t^n P_n(x) = (1 - 2xt + t^2) \sum_{n=1}^{\infty} n t^{n-1} P_n(x).$$

Equating the coefficients of t^n

$$\begin{aligned} &xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) \\ &\Rightarrow (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0. \end{aligned} \tag{6}$$

Equation (6) gives the recurrence relation between three consecutive Legendre polynomials $P_{n-1}(x), P_n(x)$ and $P_{n+1}(x)$ for all n.

Results: $P_n^{'}(1) = \frac{n(n+1)}{2}$, $P_n^{'}(-1) = (-1)^{n+1}\frac{n(n+1)}{2}$

Proof:

Since $P_n(x)$ satisfies Legendre equation, we can write

$$(1-x^2)P_n^{''}(x) - 2xP_n^{'}(x) + n(n+1)P_n(x) = 0. \tag{7}$$

Taking $x = 1$ in (7), we get $2P_n^{'}(1) = n(n+1)P_n(1)$. Since $P_n(1) = 1$, we get

$$P_n^{'}(1) = \frac{n(n+1)}{2}. \tag{8}$$

Next, by taking $x = -1$ in (7), $2P_n^{'}(-1) = -n(n+1)P_n(-1)$. Since $P_n(-1) = (-1)^n$, we get

$$P_n^{'}(-1) = (-1)^{n+1}\frac{n(n+1)}{2}. \tag{9}$$

Orthogonality of Legendre Polynomials

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases} \tag{10}$$

Proof:

Recall Legendre equation

$$(1-x^2)y^{''} - 2xy^{'} + n(n+1)y = 0. \tag{11}$$

Let $P_n(x)$ and $P_m(x)$ $(m \neq n)$ be two distinct Legendre polynomials which obviously satisfy (11).

Subsequently,

$$(1-x^2)P_n^{''}(x) - 2xP_n^{'}(x) + n(n+1)P_n(x) = 0, \tag{12}$$

$$(1-x^2)P_m^{''}(x) - 2xP_m^{'}(x) + m(m+1)P_m(x) = 0. \tag{13}$$

Multiplying (12) by $P_m(x)$ and (13) by $P_n(x)$ and subtracting

$$(1-x^2)[P_m P_n^{''} - P_n P_m^{''}] - 2x[P_m P_n^{'} - P_n P_m^{'}] + [n(n+1) - m(m+1)]P_m P_n = 0.$$

This can be written as

$$\frac{d}{dx}\left\{(1-x^2)[P_m P_n^{'} - P_n P_m^{'}]\right\} = [(m-n)(m+n+1)]P_n P_m. \tag{14}$$

Integrating both sides of (14) with respect to x from -1 to 1

$$\left\{(1-x^2)[P_m P_n^{'} - P_n P_m^{'}]\right\}_{-1}^{1} = (m-n)(m+n+1)\int_{-1}^{1} P_n(x)P_m(x)dx. \tag{15}$$

The expression on the left hand side vanishes at both limits and since $m \neq n$, we must have

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, \tag{16}$$

which proves the first part of (10).

Now for proving the second part of (10), i.e., when $m = n$, consider the following based on generating function.

Since both $P_n(x)$ and $P_m(x)$ have the same generating function, we can write

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} t^n P_n(x), \tag{17}$$

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{m=0}^{\infty} t^m P_m(x). \tag{18}$$

Multiplying (17) and (18),

$$(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m} P_n(x)P_m(x). \tag{19}$$

Integrating both sides of (19) with respect to x from -1 to 1,

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left\{ \int_{-1}^{1} P_n(x)P_m(x)dx \right\} t^{n+m}. \tag{20}$$

We know that for $m \neq n$, $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ and subsequently the double summation in (20) reduces to

$$\sum_{n=0}^{\infty} \left\{ \int_{-1}^{1} (P_n(x))^2 dx \right\} t^{2n}.$$

On the other hand

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{1-2xt+t^2} &= -\frac{1}{2t}\left[\ln(1-2xt+t^2)\right]_{-1}^{1} = \frac{1}{t}[\ln(1+t)-\ln(1-t)] \\
&= \frac{1}{t}\left\{\left(t-\frac{t^2}{2}+\frac{t^3}{3}-\cdots\right)-\left(-t-\frac{t^2}{2}-\frac{t^3}{3}-\cdots\right)\right\} \\
&= 2\left\{1+\frac{t^2}{3}+\frac{t^4}{5}+\cdots\right\} \\
&= \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}.
\end{aligned}$$

Therefore, (20) gives

$$\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}(P_n(x))^2\,dx\right\}t^{2n} = \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}. \tag{21}$$

Equating the coefficients of t^{2n} from both sides of (21),

$$\int_{-1}^{1} (P_n(x))^2 dx = \frac{2}{2n+1}. \tag{22}$$

which completes the proof of (10), i.e.,

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases} \tag{23}$$

If we write $x = \cos\theta$, the orthogonality property of Legendre polynomials can be written as

$$\int_{0}^{\pi} P_m(\cos\theta)P_n(\cos\theta)\sin\theta \; d\theta = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases} \tag{24}$$

Orthogonality property of Legendre polynomials in either form is very useful in many applications.

Legendre series

We can write the various powers of x in terms of some Legendre polynomials as follows:

$$\begin{aligned} 1 &= P_0(x),\ x = P_1(x),\ x^2 = \frac{1}{3} + \frac{2}{3}P_2(x) = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x) \\ x^3 &= \frac{3}{5}x + \frac{2}{5}P_3(x) = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x) \end{aligned}$$

Hence it follows that any third degree polynomial $p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ can be written as

$$\begin{aligned} p(x) &= b_0 P_0(x) + b_1 P_1(x) + b_2\left[\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)\right] + b_3\left[\frac{3}{5}P_1(x) + \frac{2}{5}P_3(x)\right] \\ &= \left(b_0 + \frac{b_2}{3}\right)P_0(x) + \left(b_1 + \frac{3b_3}{5}\right)P_1(x) + \frac{2b_2}{3}P_2(x) + \frac{2b_2}{5}P_3(x) \\ &= \sum_{n=0}^{3} a_n P_n(x) \end{aligned}$$

More generally, since $P_n(x)$ is a polynomial of degree n for every positive integer n, it is possible that x^n can be expressed as linear combinations of $P_0(x), P_1(x), P_2(x), \ldots, P_n(x)$ (for even n, they are $P_0(x), P_2(x), P_4(x), \ldots, P_{2n}(x)$ and for odd n, they are $P_1(x), P_3(x), P_5(x), \ldots, P_{2n-1}(x)$) so that any polynomial $p(x)$ of degree k has an expression of the form

$$p(x) = \sum_{n=0}^{K} a_n P_n(x),$$

where $K = k/2$ if k is even and $K = (k-1)/2$ if k is odd.

Actually an arbitrary function $f(x)$ can be expanded in terms of Legendre polynomials:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \tag{25}$$

which is known as Legendre series.

In order to find the unknowns a_n's, we proceed in the following way:

Multiply (25) by $P_m(x)$ and integrate term by term between -1 and 1

$$\begin{aligned}\int_{-1}^{1} f(x)P_m(x)dx &= \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)P_m(x)dx \\ &= a_m \int_{-1}^{1} \{P_m(x)\}^2 dx \\ &= \frac{2a_m}{2m+1} \text{ (due to the orthogonality of Legendre polynomials).}\end{aligned}$$

This gives us (by changing m to n)

$$a_n = (n + \frac{1}{2}) \int_{-1}^{1} f(x)P_n(x)dx. \tag{26}$$

Legendre polynomials in numerical analysis

If we recall Gaussian quadrature for computing integrals numerically, it will be interesting to note how the zeros of Legendre polynomials determine the different Gaussian quadrature formulas.

While trying to integrate a function $f(x)$ between -1 and 1, we come across the following formulas:

$$\int_{-1}^{1} f(x)dx \doteq 2f(0), \tag{27}$$

$$\int_{-1}^{1} f(x)dx \doteq f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right), \tag{28}$$

$$\int_{-1}^{1} f(x)dx \doteq \left[\frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{1}{9} f\left(\sqrt{\frac{3}{5}}\right)\right]. \tag{29}$$

We notice that the evaluation of the integrals is nothing but finding the values of the integrand at certain points, viz., $0; \pm\frac{1}{\sqrt{3}}$ and $0, \pm\sqrt{\frac{3}{5}}$, which are the zeros of the Legendre polynomials $P_1(x), P_2(x)$ and $P_3(x)$, respectively.

Similarly the zeros of higher order Legendre polynomials are used for higher order Gaussian quadrature though very infrequently.

In general

in order to evaluate $\int_a^b f(x)dx$ by Gaussian quadrature, we make the substitution $x = (b-a)\xi/2 + (b+a)/2$ so that

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f[(b-a)\xi/2 + (b+a)/2]d\xi. \quad (30)$$

The n point formula is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f[(b-a)\xi_i/2 + (b+a)/2], \quad (31)$$

where the weights w_i are given by

$$w_i = \frac{2}{(1-\xi_i^2)[P_n'(\xi_i)]^2}, \quad \text{with } \xi_i \text{ denoting a zero of } P_n(x). \quad (32)$$

Power series method is a powerful method for finding solution to some classes of second-order ordinary differential equations. Here we have discussed only Bessel functions and Legendre polynomials.

Power series method can be appropriately used to find solutions of some other special functions or orthogonal polynomials such as

- Laguerre polynomials
- Hermite polynomials
- Tchebyshev polynomials
- Modified Bessel functions
- Associated Legendre polynomials

All of above have significant importance in mathematical physics and numerical analysis.