MA 542 Differential Equations
Lecture 12
(January 28, 2022)

Orthogonal Property

If λ_n are eigenvalues of Bessel function $J_\mu(\lambda_n x)$ of a specific order μ, then

$$\int_0^1 x J_\mu(\lambda_n x) J_\mu(\lambda_m x) dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}\{J_{\mu+1}(\lambda_n)\}^2, & m = n. \end{cases} \tag{1}$$

Proof:

Let $y_1(x) = J_\mu(\lambda_n x)$ and $y_2(x) = J_\mu(\lambda_m x)$ be two solutions of Bessel's equation for positive λ_n and λ_m. Then $y_1(x)$ and $y_2(x)$ satisfy

$$y_1^{''} + \frac{1}{x} y_1^{'} + \left(\lambda_n^2 - \frac{\mu^2}{x^2}\right) y_1 = 0, \tag{2}$$

$$y_2^{''} + \frac{1}{x} y_2^{'} + \left(\lambda_m^2 - \frac{\mu^2}{x^2}\right) y_2 = 0. \tag{3}$$

Multiply (2) and (3), respectively, by $y_2(x)$ and $y_1(x)$, then subtract the results and multiply by x to get

$$\frac{d}{dx}\left[x(y_1^{'} y_2 - y_2^{'} y_1)\right] = (\lambda_m^2 - \lambda_n^2) x y_1 y_2. \tag{4}$$

Integrate (4) with respect to x from $x = 0$ to $x = 1$ to get

$$(\lambda_m^2 - \lambda_n^2) \int_0^1 x y_1 y_2 dx = |x(y_1^{'} y_2 - y_2^{'} y_1)|_0^1. \tag{5}$$

The expression on right vanishes at both $x = 0$ and $x = 1$.

Therefore if m and n are distinct, i.e., λ_n and λ_m are distinct, then

$$\int_0^1 x J_\mu(\lambda_n x) J_\mu(\lambda_m x) dx = 0, \tag{6}$$

which is the first part of (1).

Next we are required to evaluate the integral $\int_0^1 x J_\mu(\lambda_n x) J_\mu(\lambda_m x) dx$ for $m = n$.

Multiplying (2) by $2x^2y_1'$ and adjusting the terms, we get

$$\frac{d}{dx}\left(x^2{y_1'}^2\right)+\frac{d}{dx}\left(\lambda_n^2x^2y_1^2\right)-2\lambda_1^2xy_1^2-\frac{d}{dx}\left(\mu^2y_1^2\right)=0.$$

Integrate from $x=0$ to $x=1$, we get

$$2\lambda_n^2\int_0^1 xy_1^2dx=\left|x^2{y_1'}^2+(\lambda_n^2x^2-\mu^2)y_1^2\right|_0^1. \quad (7)$$

For $x=0$, the expression in brackets vanishes and since $y_1'(1)=\lambda_nJ_\mu'(\lambda_n)$, (7) gives

$$\int_0^1 x\{J_\mu(\lambda_nx)\}^2dx=\frac{1}{2}\{J_\mu'(\lambda_n)\}^2+\frac{1}{2}\left(1-\frac{\mu^2}{\lambda_n^2}\right)\{J_\mu(\lambda_n)\}^2.$$

Using some earlier relations, it results in

$$\int_0^1 x\{J_\mu(\lambda_nx)\}^2dx=\frac{1}{2}\{J_\mu'(\lambda_n)\}^2=\frac{1}{2}\{J_{\mu+1}(\lambda_n)\}^2, \quad (8)$$

which gives the second part of (1), and (6) and (8) together give the orthogonality result (1).

Another important equation which gives rise to special functions is Legendre equation:

$$(1-x^2)y^{''} - 2xy^{'} + n(n+1)y = 0. \tag{9}$$

and its solutions are known as **Legendre polynomials**.

It is obvious that $x = \pm 1$ are regular singular points of Legendre equation.

We use regular power series method (Frobenius series is not required) to find a series solution since the origin is an ordinary point in this case.

Because the region of convergence for such a series is $|x| < 1$, the infinite series will be convergent for $-1 < x < 1$.

Origin

When we try to solve Laplace's equation in spherical coordinates (r, θ, ψ), we come across three ordinary differential equations one of which (in θ) is

$$(1 - x^2)y^{''} - 2xy^{'} + \left[n(n+1) - \frac{m^2}{1 - x^2}\right] y = 0, \tag{10}$$

$y = f(x)$ is a function of θ with $x = \cos\theta$.

This equation is known as associated Legendre equation, where m and n are constants with integral values, which arise while solving Laplace's equation by separation of variables. Its solutions are known as associated Legendre polynomials, denoted by P_n^m.

When $m = 0$, equation (10) reduces to Legendre equation (9).

Assume a series solution of the form:

$$y = \sum_{k=0}^{\infty} a_k x^k \tag{11}$$

Using (11) in (9), we get

$$\sum_{k=0}^{\infty} a_k k(k-1)x^{k-2} + \sum_{k=2}^{\infty} a_{k-2}[n(n+1) - 2(k-2) - (k-2)(k-3)]x^{k-2} = 0. \tag{12}$$

Recurrence relation:

$$a_k = -\frac{[n(n+1) - 2(k-2) - (k-2)(k-3)]a_{k-2}}{k(k-1)}, \quad k \geq 2.$$

That is

$$a_k = -\frac{[(n-k+2)(n+k-1)]a_{k-2}}{k(k-1)}, \quad k \geq 2. \tag{13}$$

The coefficients can be expressed in terms of either a_0 or a_1 as follows:

$$\begin{aligned} a_2 &= -\frac{n(n+1)}{2!}a_0, \quad a_3 = -\frac{(n-1)(n+2)}{3!}a_1 \\ a_4 &= -\frac{(n-2)(n+3)}{4\cdot 3}a_2 = +\frac{n(n-2)(n+1)(n+3)}{4!}a_0 \\ a_5 &= -\frac{(n-3)(n+4)}{5\cdot 4}a_3 = +\frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1 \end{aligned}$$

A complete solution can be written in the form

$$\begin{aligned} y &= a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right] \\ &+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots\right]. \qquad (14) \end{aligned}$$

If n is even, the first series terminates after certain terms and if n is odd, the second series terminates after certain terms. In either case, the series which reduces to a finite sum is known as a Legendre polynomial or a spherical harmonic of order n, denoted by $P_n(x)$.

To obtain the standard form of Legendre polynomial, we substitute the values of a_0 and a_1 in such a way that the coefficients of the highest power of x in each series is equal to

$$\frac{(2n)!}{2^n(n!)^2}$$

These values for a_0 and a_1 are

$$\begin{aligned} a_0 &= (-1)^{n/2}\frac{n!}{2^n[(n/2)!]^2}, \\ a_1 &= (-1)^{(n-1)/2}\frac{(n+1)!}{2^n[(\frac{n-1}{2})!(\frac{n+1}{2})!]}. \end{aligned}$$

The resulting general formula is

$$P_n(x) = \sum_{k=0}^{N} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}, \tag{15}$$

where

$$N = \frac{n}{2},\ n \text{ even};\quad N = \frac{n-1}{2},\ n \text{ odd}.$$

This formula gives the first few Legendre polynomials as:

$$\begin{aligned}
P_0(x) &= 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_3(x) = \frac{1}{2}(5x^3 - 3x),\\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3),\ P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x),\\
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5),\ P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x),\\
P_8(x) &= \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35).
\end{aligned}$$

A complete solution of (9) can be written as

$$y(x) = AP_n(x) + BQ_n(x) \tag{16}$$

where the Legendre polynomial of second kind $Q_n(x)$ can be obtained from $P_n(x)$ as

$$Q_n(x) = P_n(x) \int \frac{dx}{(x^2 - 1)[P_n(x)]^2} \tag{17}$$

which is not bounded near $x = \pm 1$.

Legendre equation of order zero ($n = 0$) can be written as

$$(1 - x^2)y^{''} - 2xy^{'} = 0, \tag{18}$$

which has the compact form

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] = 0. \tag{19}$$

Integrating (19):

$$(1 - x^2)\frac{dy}{dx} = A. \tag{20}$$

Separating the variables and integrating (20):

$$\int dy = \int \frac{A}{1 - x^2} dx.$$

This gives the solution as

$$y = A\,\frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) + B. \tag{21}$$

We know that, being of second order, every Legendre equation has two linearly independent solutions with two arbitrary constants which are A and B here.

Therefore, here we denote the linearly independent solutions of (18) as

$$P_0(x) = 1, \qquad Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \tag{22}$$

It may be noted that $Q_0(x)$ diverges at $x = 1$.

As it has already been observed, for the general case of $n \neq 0$, power series solution is employed to solve Legendre equation.

Note:

In most of the cases, since solutions bounded for $x \to \pm 1$ are required, it is seen that the solution of a specific Legendre equation of order n tends to contain only $P_n(x)$.

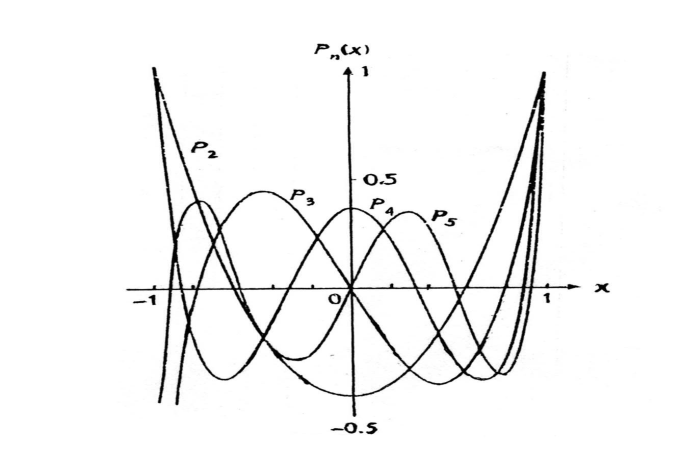

Figure 1 : Graphs of some Legendre polynomials

Legendre polynomials in terms of θ:

By substituting $x = \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, Legendre polynomial can also be expressed as

$$P_k(\cos\theta) = \sum_{n=0}^{K} \frac{(2n)!(2k-2n)!(2\cos(k-2n)\theta)}{2^{2k}(n!)^2((k-n)!)^2}. \tag{23}$$

where

$$K = \frac{k}{2},\ k \text{ even};\quad K = \frac{k-1}{2},\ k \text{ odd}.$$

The above expression is required for some problems in which it becomes essential that Legendre polynomial be expressed in terms of θ. For such problems, usually in three-dimensional spherical problems, the solutions remain in terms of θ.

Rodrigues' Formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n, \; n = 0, 1, 2, 3, \ldots$$

Proof:

The binomial expansion of $(x^2-1)^n$ is given by

$$(x^2-1)^n = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!k!} x^{2n-2k}.$$

Differentiating it n times gives

$$\frac{d^n}{dx^n}(x^2-1)^n = \sum_{k=0}^{N} (-1)^k \frac{n!(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}, \tag{24}$$

where the last term is a constant and $N = n/2$ when n is even and $N = (n-1)/2$ when n is odd.

Now on comparing (15) and (24), the desired result can be obtained.