MA 542 Differential Equations

Lecture 11

(January 27, 2022)

Important recurrence relations:

$$\frac{d}{dx}\left(x^{\mu}J_{\mu}(x)\right) = x^{\mu}J_{\mu-1}(x), \quad (1)$$

$$\frac{d}{dx}\left(x^{-\mu}J_{\mu}(x)\right) = -x^{-\mu}J_{\mu+1}(x). \quad (2)$$

Recall $\quad J_{\mu}(x) = \sum_{k=0}^{\infty}(-1)^{k}\frac{(x/2)^{2k+\mu}}{k!\,\Gamma(\mu+k+1)}.$

Proof of (1):

$$\begin{aligned}
\frac{d}{dx}\left(x^{\mu}J_{\mu}(x)\right) &= \frac{d}{dx}\left(\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k+2\mu}}{2^{2k+\mu}k!\,\Gamma(\mu+k+1)}\right)\\
&= \sum_{k=0}^{\infty}(-1)^{k}\frac{2(k+\mu)x^{2k+2\mu-1}}{2^{2k+\mu}k!\,\Gamma(\mu+k+1)}\\
&= \sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k+2\mu-1}}{2^{2k+(\mu-1)}k!\,\Gamma(\mu+k)}\\
&= x^{\mu}\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k+(\mu-1)}}{2^{2k+(\mu-1)}k!\,\Gamma((\mu-1)+k+1)} = x^{\mu}J_{\mu-1}(x).
\end{aligned}$$

In a similar manner, the result (2) can be obtained.

Opening up the differentials in equations (1) and (2) and dividing them by $x^{\pm\mu}$, respectively, the following important results can be obtained:

$$J_{\mu-1}(x) + J_{\mu+1}(x) = \frac{2\mu}{x} J_\mu(x), \tag{3}$$

$$J_{\mu-1}(x) - J_{\mu+1}(x) = 2J_\mu^{'}(x). \tag{4}$$

The above relations show the connection between some specific Bessel functions.

Specifically, for $\mu = n$,

Result (3) can be interpreted as the recursion relation between three consecutive Bessel functions of integer order whereas (4) shows that the derivative of an integer order Bessel function can be expressed in terms of that Bessel function and the following integer order Bessel function.

We know

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Then, from (3), the following can be obtained:

For $\mu = \pm(1/2)$

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right),$$

$$J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\frac{\cos x}{x} - \sin x \right).$$

For $\mu = \pm(3/2)$

$$J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right),$$

$$J_{-5/2}(x) = -\frac{3}{x}J_{-3/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x\right).$$

- Such calculation can be continued indefinitely and hence every Bessel function $J_{m+1/2}(x)$, with m as an integer, is elementary.
- Liouville proved that these are the only cases in which $J_\mu(x)$ is elementary. For all other values of μ, they are not elementary.

Recall from (1) and (2) that

$$\frac{d}{dx}\left(x^{\mu} J_{\mu}(x)\right) = x^{\mu} J_{\mu-1}(x),$$
$$\frac{d}{dx}\left(x^{-\mu} J_{\mu}(x)\right) = -x^{-\mu} J_{\mu+1}(x).$$

Integrating both sides, we get

$$\int x^{\mu} J_{\mu-1}(x) dx = x^{\mu} J_{\mu}(x) + c_1,$$
$$\int x^{-\mu} J_{\mu+1}(x) dx = -x^{-\mu} J_{\mu}(x) + c_2.$$

This shows that

Integration of expressions containing Bessel functions yields results in terms of some other Bessel functions.

One simple example is (for $\mu = 0$ and by using $J_{-1}(x) = (-1)J_1(x)$)

$$J_0(x) = -\int J_1(x)dx + c.$$

Another example is (for $\mu = 1$)

$$xJ_1(x) = \int xJ_0(x)dx + c.$$

In a similar manner, we can obtain

$$\int_0^x x^2 J_0(x)dx = x^2 J_1(x) + xJ_0(x) - \int_0^x J_0(x)dx,$$
$$\int_0^x x^3 J_0(x)dx = x^3 J_1(x) - 2x^2 J_2(x).$$

Observation from above two:

The integral $\int_0^x x^{m+n} J_0(x)dx$ can be reduced and evaluated completely when $m+n$ is an odd positive integer.However, the reduction is terminated by a term $\int_0^x J_0(x)dx$ if $m+n$ is even.

In order to evaluate the integral $\int_0^x J_0(x)dx$, we repeatedly integrate $J_{\mu-1}(x) - J_{\mu+1}(x) = 2J_{\mu}^{'}(x)$ for $\mu = 1, 3, \ldots$:

$$\int_0^x J_0(x)dx - \int_0^x J_2(x)dx = 2J_1(x),$$
$$\int_0^x J_2(x)dx - \int_0^x J_4(x)dx = 2J_3(x),$$

and so on.

Adding both sides separately

$$\int_0^x J_0(x)dx = 2\{J_1(x) + J_3(x) + \cdots\} = 2\sum_{n=0}^{\infty} J_{2n+1}(x).$$

Definition of generating function:

Let $\{a_n\}_{n\geq 0}$ be a sequence of numbers. The generating function associated with this sequence is the series $G(x) = \sum_{n\geq 0} a_n x^n$.

In other words, the above series can be assumed to be generated by the function $G(x)$.

Consider the following and observe the generating functions:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x},$$

$$\sum_{n=0}^{\infty} (ax)^n = \frac{1}{1-ax},$$

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}.$$

The generating function of the Bessel functions is

$$G(x,t) = \exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\}. \tag{5}$$

This function can be developed into a Laurent series. The coefficient of t^n in the expansion is the Bessel function of argument x and order n.

We can write

$$\exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} t^n J_n(x). \tag{6}$$

(6) can be obtained by considering the series expansions for $\exp(xt/2)$ and $\exp(-x/(2t))$, multiplying those series and then comparing the coefficients of t^n.

Further recall that

$$J_{-n}(x) = (-1)^n J_n(x). \tag{7}$$

Using (7), we can rewrite (6) as

$$\exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\} = J_0(x) + \sum_{n=1}^{\infty}\left[t^n + (-1)^n t^{-n}\right] J_n(x). \tag{8}$$

Let $t = \exp(\mathrm{i}\theta)$ so that $1/2(t - 1/t) = \dfrac{\exp(\mathrm{i}\theta) - \exp(-\mathrm{i}\theta)}{2} = \mathrm{i}\sin\theta$.

Then (8) gives

$$\exp(\mathrm{i}x\sin\theta) = J_0(x) + \sum_{n=1}^{\infty}\left[\exp(\mathrm{i}n\theta) + (-1)^n\exp(-\mathrm{i}n\theta)\right] J_n(x). \tag{9}$$

We get

$$t^n + (-1)^n t^{-n} = \left[\exp(\mathrm{i}n\theta) + (-1)^n\exp(-\mathrm{i}n\theta)\right] = 2\cos(2k\theta), \quad n = 2k, \tag{10}$$

$$t^n + (-1)^n t^{-n} = \left[\exp(\mathrm{i}n\theta) + (-1)^n\exp(-\mathrm{i}n\theta)\right] = 2\mathrm{i}\sin((2k-1)\theta), \quad n = 2k-1. \tag{11}$$

Therefore, (9) can be written as

$$\exp(\mathrm{i}x\sin\theta) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos(2k\theta) + 2\mathrm{i}\sum_{k=1}^{\infty} J_{2k-1}(x)\sin((2k-1)\theta). \tag{12}$$

Equating real and imaginary parts

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos(2k\theta), \tag{13}$$

$$\sin(x\sin\theta) = 2\sum_{k=1}^{\infty} J_{2k-1}(x)\sin((2k-1)\theta). \tag{14}$$

Now multiply both sides of (13) by $\cos n\theta$ and both sides of (14) by $\sin n\theta$ and integrating each with respect to θ from 0 to π

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta d\theta = \begin{cases} \pi J_n(x), & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{15}$$

$$\int_0^{\pi} \sin(x\sin\theta)\sin n\theta d\theta = \begin{cases} 0, & n \text{ even}, \\ \pi J_n(x), & n \text{ odd}. \end{cases} \tag{16}$$

Adding (15) and (16), we can get for all positive integral values of n

$$\int_0^{\pi} [\cos(x \sin\theta)\cos n\theta + \sin(x \sin\theta)\sin n\theta]\, d\theta = \pi J_n(x). \tag{17}$$

Simplification gives

$$\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta = \pi J_n(x). \tag{18}$$

Further, putting $\theta = \pi/2$ in (13) and (14), we get, respectively,

$$\cos x = J_0(x) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(x), \tag{19}$$

$$\sin x = 2\sum_{k=1}^{\infty} (-1)^{k-1} J_{2k-1}(x). \tag{20}$$

The above show a relationship between circular functions and Bessel functions of first kind.

Consider a boundary-value problem which consists of

1. a second-order homogeneous linear differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0 \tag{21}$$

   where p, q and r are real functions such that p has a continuous derivative, q and r are continuous and $p(x) > 0$ and $r(x) > 0$ for all x on a real interval $a \leq x \leq b$, and λ is a parameter independent of x;

2. two supplementary conditions

$$A_1 y(a) + A_2 y'(a) = 0 \tag{22a}$$

$$B_1 y(b) + B_2 y'(b) = 0 \tag{22b}$$

   where A_1, A_2, B_1 and B_2 are real constants such that A_1 and A_2 are not both zero and B_1 and B_2 are not both zero.

This is called the Sturm-Liouville Problem.

The function $r(x)$ is called the weight function of the differential equation.

For Bessel's equation,

$r(x) = x$.

Definition

Consider the Sturm-Liouville problem consisting of the differential equation (21) and the supplementary conditions (22). The values of the parameter λ in (21) for which there exist non-trivial solutions of the problem are called the characteristic values or eigenvalues of the problem. The corresponding non-trivial solutions are called the characteristic functions or eigenfunctions of the problem.

For Sturm-Liouville problem

1. There exists an infinite number of characteristic values λ_n of the eigen problem. These characteristic values λ_n can be arranged in a monotonic increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

such that $\lambda_n \to +\infty$ as $n \to \infty$.

2. Corresponding to each characteristic value λ_n there exists a one-parameter family of characteristic functions ϕ_n. Each of these characteristic functions is defined on $a \leq x \leq b$, and any two characteristic functions corresponding to the same characteristic value are nonzero constant multiples of each other.

3. Each characteristic function ϕ_n corresponding to the characteristic value λ_n $(n = 1, 2, 3, \ldots)$ has exactly $(n-1)$ zeros in the open interval $a < x < b$.

Orthogonality of characteristic functions

Two functions f and g are called orthogonal with respect to the weight function r on the interval $a \leq x \leq b$ if and only if

$$\int_a^b r(x)f(x)g(x)\, dx = 0.$$

Definition

Let $\{\phi_n\}, n = 1, 2, 3, \ldots$ be an infinite set of functions defined on the interval $a \leq x \leq b$. The set $\{\phi_n\}$ is called an orthogonal system with respect to the weight function r on $a \leq x \leq b$ if every two distinct functions of the set are orthogonal with respect to r on $a \leq x \leq b$. That is, the set $\{\phi_n\}$ is orthogonal with respect to r on $a \leq x \leq b$ if

$$\int_a^b r(x)\phi_m(x)\phi_n(x)\, dx = 0, \quad \text{for} \;\; m \neq n.$$

Theorem

Let λ_m and λ_n be any two distinct characteristic values of a Sturm-Liouville problem. Let ϕ_m and ϕ_n, respectively, be the characteristic functions corresponding to λ_m and λ_n. Then the characteristic functions ϕ_m and ϕ_n are orthogonal with respect to the weight function r on the interval $a \leq x \leq b$.

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