MA 542 Differential Equations

Lecture 10

(January 25, 2022)

The differential equation

$$x^2y^{''} + xy^{'} + (x^2 - \mu^2)y = 0, \tag{1}$$

where μ is a constant, is called Bessel's equation of order μ and its solutions are known as Bessel functions of order μ. (Unless stated, μ is not an integer)

Laplace's equation in cylindrical coordinates (r, θ, z) is given by

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} + u_{zz} = 0. \tag{2}$$

In order to solve this equation, we can use the separation of variables method by assuming a solution of (2) of the form

$$u(r, \theta, z) = \mathcal{R}(r)\mathcal{T}(\theta)\mathcal{Z}(z). \tag{3}$$

Using (3) in (2), we see that the partial differential equation (2) is converted to three ordinary differential equations – one each in r, θ and z as follows:

$$r^2\mathcal{R}^{''} + r\mathcal{R}^{'} + (\lambda^2 r^2 - \mu^2) = 0, \quad \mathcal{T}^{''} + \mu^2\mathcal{T} = 0, \quad \mathcal{Z}^{''} - \lambda^2\mathcal{Z} = 0,$$

where the constants λ and μ are separation constants.

The last two equations of the above system give rise to two simple solutions whereas the first one is not known to have some standard function(s) as solutions. This equation is nothing but what we call Bessel's equation of order μ with parameter λ.

We can easily see that $x = 0$ is a singular point of equation (1). Moreover, here $P(x) = 1/x,\ Q(x) = (x^2 - \mu^2)/x^2$. Hence $xP(x) = 1$ and $x^2Q(x) = x^2 - \mu^2$ which show that $x = 0$ is a regular singular point. Let's assume the solution of (1) to be of the form (Frobenius series)

$$y = x^m \sum_{k=0}^{\infty} a_k x^k.$$

Indicial equation gives $m = \pm\mu$. First consider $m = +\mu$. **($\mu = \pm 1/2$ is special case to be discussed later.)**

Recurrence relation (Equating the coefficient of x^{m+k})

$$a_k = -\frac{a_{k-2}}{k(k+2\mu)}, \quad k \geq 2. \tag{4}$$

We know that $a_0 \neq 0$ and $a_1 = 0$ above since $a_{-1} = 0$.

It results in

$$a_1 = a_3 = a_5 = \cdots = a_{2k-1} = \cdots = 0.$$

All other coefficients can be expressed as

$$\begin{aligned}
a_2 &= -\frac{a_0}{2(2\mu+2)} = -\frac{a_0}{2^2(\mu+1)}, \\
a_4 &= -\frac{a_2}{4(2\mu+4)} = +\frac{a_0}{2^2 2!(\mu+1)(\mu+2)}, \\
a_6 &= -\frac{a_4}{6(2\mu+6)} = -\frac{a_0}{2^3 3!(\mu+1)(\mu+2)(\mu+3)}.
\end{aligned}$$

This gives rise to the general coefficient as

$$a_{2k} = (-1)^k \frac{a_0}{2^{2k} k!\,(\mu+1)(\mu+2)\cdots(\mu+k)}.$$

Hence a solution can be written as

$$y = a_0 x^{\mu} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k} k!\,(\mu+1)(\mu+2)\cdots(\mu+k)}. \tag{5}$$

Bessel function of first kind of order μ, denoted by $J_\mu(x)$, is defined by putting $a_0 = 1/(2^{\mu}\Gamma(\mu+1))$ in (5) so that

$$y \equiv J_\mu(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k+\mu}}{k!\,\Gamma(\mu+k+1)} \tag{6}$$

This is one of the solutions of equation (1) and the other linearly independent solution can be obtained by considering $m = -\mu$.

That is,

$$J_{-\mu}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k-\mu}}{k!\,\Gamma(-\mu+k+1)}. \tag{7}$$

Hence the general solution can be written as (μ is a fraction)

$$y = AJ_{\mu}(x) + BJ_{-\mu}(x), \tag{8}$$

where A and B are arbitrary constants.

When $\mu = n$, i.e., an integer

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k+n}}{k!\,(n+k)!}, \tag{9}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k-n}}{k!\,(k-n)!}. \tag{10}$$

However, $J_{\mu}(x)$ and $J_{-\mu}(x)$ are not linearly independent when μ takes integer values, i.e, $\mu = n$.

In fact $J_n(x)$ and $J_{-n}(x)$ are related by

$$J_{-n}(x) = (-1)^n J_n(x), \tag{11}$$

and hence they are not linearly independent.

Proof:

$$\begin{aligned}
J_{-n}(x) &= \sum_{k=0}^{\infty}(-1)^k \frac{(x/2)^{2k-n}}{k!\,(k-n)!} \\
&= \sum_{k=0}^{n-1}(-1)^k \frac{(x/2)^{2k-n}}{k!\,(k-n)!} + \sum_{k=n}^{\infty}(-1)^k \frac{(x/2)^{2k-n}}{k!\,(k-n)!} \\
&= \sum_{k=n}^{\infty}(-1)^k \frac{(x/2)^{2k-n}}{k!\,(k-n)!} \quad \text{(replacing } k \text{ by } n+M\text{)} \\
&= \sum_{M=0}^{\infty}(-1)^{M+n} \frac{(x/2)^{-n+2n+2M}}{(n+M)!\,M!} \\
&= (-1)^n \sum_{M=0}^{\infty}(-1)^M \frac{(x/2)^{n+2M}}{(n+M)!\,M!} = (-1)^n J_n(x).
\end{aligned}$$

For $\mu = n$, the second linearly independent solution is found as

$$Y_n(x) = \lim_{\mu \to n} Y_\mu(x) = \frac{\cos \pi\mu J_\mu(x) - J_{-\mu}(x)}{\sin \pi\mu}, \tag{12}$$

which is known as Bessel function of second kind. $Y_\mu(x)$ has an important property: $Y_\mu(x) \to -\infty$ when $x \to 0$.

In that case, for $\mu = n$, the general solution of (1) can be written as

$$y = AJ_n(x) + BY_n(x). \tag{13}$$

The most useful Bessel functions are the ones of order 0 and 1:

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots, \tag{14}$$

$$J_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+1)!} \left(\frac{x}{2}\right)^{2k+1} = \frac{x}{2} - \frac{1}{1!2!} \left(\frac{x}{2}\right)^3 + \frac{1}{2!3!} \left(\frac{x}{2}\right)^5 + \cdots \tag{15}$$

An interesting result follows from (14) and (15):

$$J_0'(x) = -J_1(x) \tag{16}$$

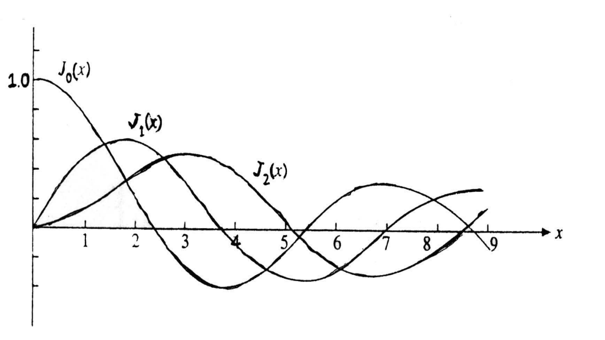

Figure 1 : Graphs of $J_0(x)$, $J_1(x)$ and $J_2(x)$

Observations

- Each curve has a damped oscillatory behaviour producing an infinite number of positive zeros.
- These zeros occur alternately − similarly as for $\cos x$ and $\sin x$.
- There exist two relations: $J_0'(x) = -J_1(x)$ and $(\cos x)' = -\sin x$.

Another observation

$J_0(x)$ begins with a constant term (which is 1) whereas $J_1(x)$ begins with a term containing x. Similarly $J_2(x), J_3(x), \ldots$ etc. begin with a term containing $x^2, x^3, \ldots$, respectively.

Therefore, for integer order Bessel functions,

$$J_n(0) = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases}$$

In some applied problems, another form of general solution is used:

$$y(x) = AH_\mu^{(1)}(x) + BH_\mu^{(2)}(x). \tag{17}$$

$H_\mu^{(1)}(x)$ and $H_\mu^{(2)}(x)$ are, respectively, known as Hankel functions of first and second kind of order μ, or Bessel function of third kind of order μ.

They are defined by the relations

$$H_\mu^{(1)}(x) = J_\mu(x) + \mathrm{i}Y_\mu(x), \tag{18}$$

$$H_\mu^{(2)}(x) = J_\mu(x) - \mathrm{i}Y_\mu(x). \tag{19}$$

Principal asymptotic forms of Hankel functions:

For fixed μ and $r \to \infty$

$$H_\mu^{(1)}(r) \sim \sqrt{2/\pi r}\ \exp\{\mathrm{i}(r - \frac{1}{2}\mu\pi - \frac{1}{4}\pi)\}, \quad -\pi < \mathrm{Arg}\ r < 2\pi, \tag{20}$$

$$H_\mu^{(2)}(r) \sim \sqrt{2/\pi r}\ \exp\{-\mathrm{i}(r - \frac{1}{2}\mu\pi - \frac{1}{4}\pi)\}, \quad -2\pi < \mathrm{Arg}\ r < \pi. \tag{21}$$

Consider Bessel's equation

$$x^2y'' + xy' + (x^2 - \mu^2)y = 0. \tag{22}$$

Use the substitution $u(x) = \sqrt{x}y(x)$ so that

$$y(x) = x^{-1/2}u.$$

Subsequently, (22) gets transformed to

$$\frac{d^2u}{dx^2} + \left(1 + \frac{1-4\mu^2}{4x^2}\right)u = 0. \tag{23}$$

When x is very large, equation (23) closely approximates the equation

$$\frac{d^2u}{dx^2} + u = 0. \tag{24}$$

which has $u_1(x) = \cos x$ and $u_2(x) = \sin x$ as the two linearly independent solutions.

It is fair to expect that for large values of x, any Bessel function $y(x)$ will behave like some linear combination of

$$\frac{\cos x}{\sqrt{x}} \quad \text{and} \quad \frac{\sin x}{\sqrt{x}}.$$

It is actually supported by

$$J_\mu(x) = \frac{\sqrt{2}}{\pi x} \cos\left(x - \frac{\pi}{4} - \frac{\mu\pi}{2}\right) + \frac{r_1(x)}{x^{3/2}},$$
$$Y_\mu(x) = \frac{\sqrt{2}}{\pi x} \sin\left(x - \frac{\pi}{4} - \frac{\mu\pi}{2}\right) + \frac{r_2(x)}{x^{3/2}},$$

where $r_1(x)$ and $r_2(x)$ are bounded as $x \to \infty$.

It can also be clearly observed that equation (24) can be obtained from equation (23) by putting $\mu = \pm(1/2)$.

In other words,

for this case, equation (23) shows that general solution $y(x)$ of converted Bessel's equation can be expressed in terms of two forms (depending on whether we consider large x or $\mu = \pm 1/2$):

$$y = \frac{1}{\sqrt{x}} \left(c_1 \cos x + c_2 \sin x \right), \tag{25}$$

$$y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x). \tag{26}$$

Therefore it must be true that

$$\sqrt{x} J_{1/2}(x) = a \cos x + b \sin x,$$
$$\sqrt{x} J_{-1/2}(x) = c \cos x + d \sin x,$$

for certain constants a, b, c and d.

By using the series solution expressions for these Bessel functions and trigonometric functions, it can be obtained that $a = 0$, $b = \sqrt{(2/\pi)}$, $c = \sqrt{(2/\pi)}$ and $d = 0$.

Therefore,

$J_{1/2}(x)$ and $J_{-1/2}(x)$ are connected to $\sin x$ and $\cos x$ by

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \tag{27}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \tag{28}$$