MA 542 Differential Equations
Lecture 1
(January 6, 2022)

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Some Beautiful Quotes

"The deep study of nature is the most fruitful source of mathematical discoveries. By offering to research a definite end, this study has the advantage of excluding vague questions and useless calculations; besides it is a sure means of forming analysis itself and of discovering the elements which it most concerns us to know, and which natural science ought always to conserve."
-Jean Baptiste Joseph Fourier

"My powers are ordinary. Only my application brings me success." - Sir Isaac Newton

"The enchanting charms of this sublime science reveal only to those who have the courage to go deeply into it." -Carl Friedrich Gauss

"Nature always tends to act in the simplest way." -Daniel Bernoulli

"What we know is not much. What we don't know is enormous." -Pierre Simon De Laplace

"For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear."
-Leonhard Euler

Tea/Coffee

Figure 1 : Hot Tea

Figure 2 : Cold Tea

Falling Body

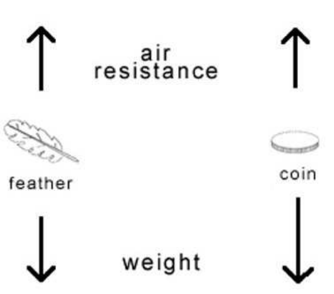

Figure 3 : Falling Body

Pendulum

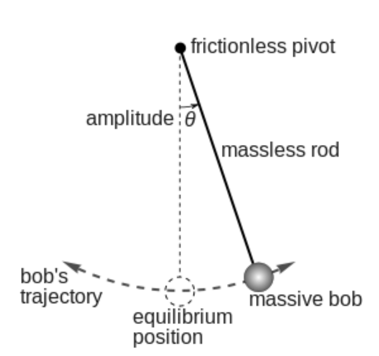

Figure 4 : Motion of a pendulum

Figure 5 : Sloshing

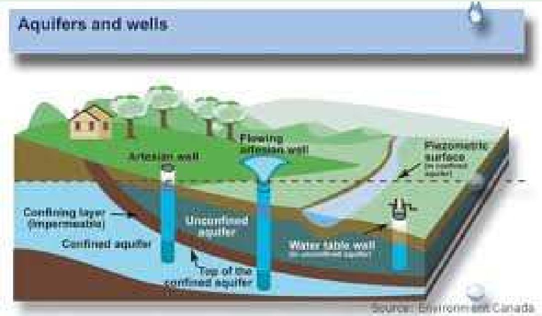

Figure 6 : Aquifer

Many of the general laws of nature – in physics, chemistry, biology, astronomy and what not–

find their most natural expression in differential equations.

Applications are mainly in the areas of mathematics itself, other branches of science, engineering, economics and many other fields of applied sciences. Why is it so?

Consider

- Planets, meteors
- Rainbows, clouds, hurricanes
- Oceans, lakes, rivers

Astronomy is probably the first discipline which extensively used mathematics (differential equations) and showed how important they are to know our universe. Later on, differential equations played a paramount role in the projection of a satellite into its orbit.

To understand atmosphere and the universe, we need to know physics as well mathematics very comprehensively.

Figure 7 : River in flow

Figure 8 : Lakes

Figure 9 : Niagara Falls

"Newton has shown us that a law is only a necessary relation between the present state of the world and its immediately subsequent state. All the other laws since discovered are nothing else; they are in sum, differential equations." -Henri Poincaré

"The integrals which we have obtained are not only general expressions which satisfy the differential equation, they represent in the most distinct manner the natural effect which is the object of the phenomenon when this condition is fulfilled, the integral is, properly speaking, the equation of the phenomenon; it expresses clearly the character and progress of it, in the same manner as the finite equation of a line or curved surface makes known all the properties of those forms." -Jean Baptiste Joseph Fourier

We know that if $y = f(x)$ or $y = f(t)$ is a given function,

then its derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}y}{\mathrm{d}t}$ can be interpreted as the rate of change of y with respect to x or t.

In most of the natural processes,

the variables involved and their rates of changes are connected to one another by means of the basic scientific principles that govern the process.

When this connection is expressed in mathematical symbols, the result is quite often a differential equation or a system of differential equations. Let us consider some examples we are already familiar with.

Example 1

According to Newton's second law of motion,

the acceleration a of a body of mass m is proportional to the total force F acting on it, with $1/m$ as the constant of proportionality, so that
$a = F/m$ or

$$ma = F. \tag{1}$$

Suppose, for instance, that

a body of mass m falls freely under the action of gravity alone, then the only force acting on it is mg.

If y is the distance down to the body from some fixed height,

then its velocity $v = \dfrac{\mathrm{d}y}{\mathrm{d}t}$ is the rate of change of position and its acceleration $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}$ is the rate of change of velocity.

With this notation, equation (1) becomes

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = mg,$$

or,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = g. \tag{2}$$

If we change the situation by assuming that there is an air resistance proportional to the velocity,

then the total force acting on the body is $mg - k(\mathrm{d}y/\mathrm{d}t)$.

Subsequently, equation (1) takes the form

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = mg - k\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{3}$$

Equations (2) and (3) are the differential equations that express the essential attributes of two physical processes of a similar problem under consideration.

They are, respectively, called undamped motion and damped motion of the body.

Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional

to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings).

Newton's this law makes a statement about an instantaneous rate of change of the temperature.

When we translate this verbal statement into mathematical symbols,

we arrive at a simple first-order ordinary differential equation.

The solution to this equation will then be a function that tracks the complete record of the temperature of the object over all time.

If T is the temperature of an object at time t and S is the temperature of its surroundings, then this law formulates into

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - S), \tag{4}$$

where k is a constant of proportionality.

If T_0 is the initial temperature,

the temperature of the object at any time t is given by

$$T(t) = S + (T_0 - S)e^{-kt}. \tag{5}$$

Equation (4) represents cooling.

If the problem has to be a heating problem, then the minus sign in (4) gets replaced by a plus sign.

To have a specific value of k, we need to derive another condition.

Consider a pendulum of length l whose bob has mass m.

Then the equation of motion (undamped case) is given by

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin\theta = 0.$$

Is this the equation we usually know?

Or the equation we know is different from this?

The applicable form is the following linearized version:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\theta = 0.$$

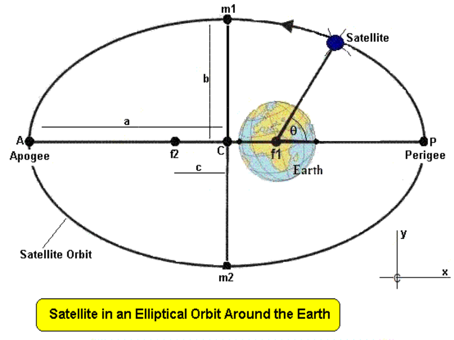

Figure 10 : An orbiting satellite

A system of differential equations

Consider the rectangular coordinate system $Oxyz$ with the origin at the center of mass of a planet.

The xy-plane coincides with the equatorial plane.

The z-axis is directed along the planet's rotation axis.

The system of equations of motion of a satellite of the planet:

$$\begin{aligned}
\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \frac{\partial W}{\partial x} &= \frac{\partial R}{\partial x}, \\
\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\partial W}{\partial y} &= \frac{\partial R}{\partial y}, \\
\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} - \frac{\partial W}{\partial z} &= \frac{\partial R}{\partial z},
\end{aligned}$$

where

R is the distance between the planet and the satellite and

$$W = \frac{gm}{2}\left\{\frac{1+\mathrm{i}\sigma}{r_1} + \frac{1-\mathrm{i}\sigma}{r_2}\right\},$$

in which

$$r_1 = \sqrt{x^2 + y^2 + [z - c(\sigma + \mathrm{i})]^2},$$
$$r_2 = \sqrt{x^2 + y^2 + [z - c(\sigma - \mathrm{i})]^2},$$

g is the gravitational constant, m is the mass of the planet, c and σ are certain constants, $\mathrm{i} = \sqrt{-1}$.

Extensive use of differential equations is found nowadays in topics such as

- Drug delivery
- Growth/retardation of cells
- Epidemic/pandemic models (including COVID-19)

Governing equation:

Every modeling has its own appropriate governing differential equation or a system of differential equations.

Initial data:

Every modeling has its own appropriate initial data which are used as initial condition(s).

Differential equations play a pivotal role in modeling and solving various problems in river mechanics such as

- Flood routing
- Flow discharge
- Dambreak flow
- River-bed and River-bank erosion
- Sediment transport

Works

They may be analytical, computational or experimental.

Initial data:

Every modeling has its own appropriate initial data which are used as initial condition(s).

Benefits:

To take appropriate steps to stop/reduce/control erosion, sediment accumulation etc.

(a) River

(b) Canal

Figure 11 : River flows

We have already discussed Newton's law of cooling and Kolmogorov Predator-Prey model.

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

states that the rate of change of a quantity is proportional to the quantity itself. if $k > 0$ and $y > 0$, then it ia problem of growth whereas for $k < 0$ and $y > 0$, it is a problem of decay.
y can be considered to the population of bacteria and the solution gives the total number of bacteria given the initial number.
The solution $y = ce^{kt}$ clearly shows that the growth is exponential.

The same equation (with a negative sign) can be applied for a problem of decay of radioactive substances with the information that half-life of radium is 1600 years.

Consider application to chemical species reaction

A first-order chemical reaction is defined when the rate of change of concentration of the chemical species is proportional to the concentration. If a is the concentration of species A at time $t = 0$ and x is the concentration at time t, then a first-order reaction is defined by a differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(a - x).$$

A second-order chemical reaction or bimolecular reaction can be defined when molecules of species A and B react to form molecules of species C. If original concentration of A and B are, respectively, a and b and x and x is the concentration at time t, then the second-order reaction can be expressed as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(a - x)(b - x).$$

Let us now come to electric circuits.

Let us consider an electric circuit such that

- Q is the instantaneous charge
- I is the instantaneous current
- E is the instantaneous emf
- R is the constant resistance
- C is the constant capacitance

Current is proportional to emf, i.e.,

$$I \propto E,$$

giving

$$E = RI.$$

Instantaneous current is given by

$$I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t}.$$

giving

$$E = RI.$$

Let us now take up differential equation related to electric circuits.

The algebraic sum of all the instantaneous voltage drops around any closed loop is zero, or the voltage supplied is equal to the sum of the voltage drops. This was developed Gustav Robert Kirchoff, a German physicist.

The following hold:

- voltage drop across resistance $= RI$
- voltage drop across inductance $= L\frac{\mathrm{d}I}{\mathrm{d}t}$
- voltage applied $= E(t)$

Kirchoff's law gives the differential equation for R-L circuit

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI = E(t).$$

with some initial condition, say $I(0) = 0$, i.e., current is zero at time $t = 0$.

EMF can be of two types:

- $E(t) = E_0$, having a constant voltage
- $E(t) = E_0 \cos \omega t$, having a periodic voltage instead of a constant voltage.

Second-order equations have immense significance from practical point of view.

- Many physical phenomena can be represented in terms of differential equations
- In many aspects of mathematical physics

Most general form:

$$y''(x) = f(x, y, y')$$

Specific form:

$$y'' + P(x)y' + Q(x)y = R(x).$$

If $R(x) \equiv 0$, equation is called homogeneous, otherwise non-homogeneous.

Solutions depend on the functions:

- $P(x)$ and $Q(x)$ are constants.
- $P(x)$ and $Q(x)$ have specific form in x.
- $P(x)$ and $Q(x)$ have arbitrary form in x.

We have already discussed Newton's second law of motion and also the motion of a pendulum bob.

Spring mass system without external force

The equation of motion of a particle of mass m, attracted to a fixed point in its line of motion by a force of c times its distance from the point, and damped by a frictional resistance of k times its velocity gives the differential equation

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + k\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0.$$

Spring mass system with external force

With specific values of the constants and consideration of the action of an external periodic force, the equation of motion gets modified to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}y}{\mathrm{d}t} + p^2 y = a\cos qt.$$

Kirchoff's law

The earlier equation for $I(t)$ can be expressed as a second-order equation for $Q(t)$:

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = E.$$

Beam equation

A beam of length $2L$ is subjected to a vertical load $W(x)$ such that x denotes the distance from one end of the beam. Then the deflection $y(x)$ satisfies

$$EI\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = W(x), \qquad 0 < x < 2L$$

where

- E: the modulus of elasticity
- I: moment of inertia.

Both are assumed constant.

We have discussed only a very few examples which may be considered as some drops of water in an ocean of physical problems for which ordinary differential equations play a huge role.

The wave propagation and its application is encountered in many areas of physical interest

Studies on waves have received focus in science and engineering, and water waves serve as important models for investigation.

Applications:

The study of different kinds of water waves is of importance for various applications. For example,

1. for predicting the behaviour of floating structures (immersed totally or partially) such as tension-leg platform, buoys, ships, submarines,
2. for describing flows over bottom topography.
3. for installing appropriate structures as breakwater to reduce wave impact.
4. for analyzing scattering and trapping of waves
5. for finding ocean space for structures usually installed on land.

Figure 12 : Water Waves

Water Waves

(a) Oil rig, The South China Sea

(b) Sea-cage, Taiwan

Figure 13 : Structures in ocean

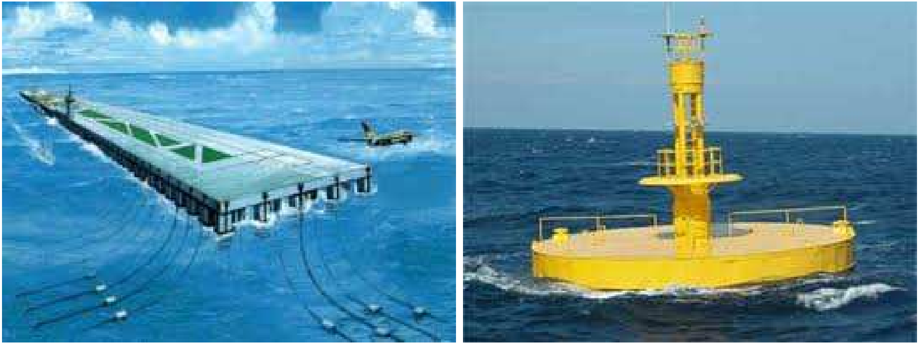

Figure 14 : Floating structures in ocean

Figure 15 : Ship in motion in ocean

Figure 16 : Oil spill

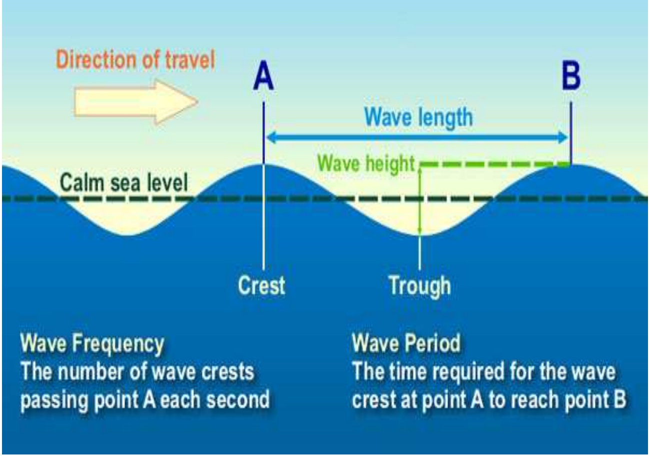

Figure 17 : Schematic diagram of water wave

Essential notations:

- $k (= 2\pi/L)$ - wavenumber
- L - wavelength
- T - time period
- σ or ω - angular wave frequency
- crest - the highest point of wave
- trough - the lowest point of wave
- η - instantaneous wave elevation (above mean free surface)
- A - wave amplitude
- Φ - time-dependent velocity potential
- ϕ - time-independent velocity potential

Two theories available:

- Linear water wave theory (Small amplitude - amplitude much smaller than wavelength)
- Nonlinear water wave theory (finite amplitude)

Governing equation

$$\nabla^2\phi = 0 \qquad \text{(Laplace's equation)},$$

or,

$$(\nabla^2 - \nu^2)\phi = 0 \qquad \text{(Modified Helmholtz equation)}.$$

is the equation of continuity for incompressible fluid.

Boundary conditions:

- At the sea/ocean bottom
- At the upper surface (may be free surface or any other surface)
- At any vertical boundary (problem specific)
- At the body surface, if any.

On the free surface $z = \eta$, the kinematic boundary condition (in general)

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial \Phi}{\partial y}, \quad \text{at} \quad z = \eta(x, y, t). \tag{6}$$

Linearized dynamic boundary condition

$$\frac{\partial \Phi}{\partial t} + \frac{p}{\rho} + g\eta = 0, \quad \text{at} \quad z = 0, \tag{7}$$

Linearized kinematic condition

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0. \tag{8}$$

Combined free surface condition at $z = 0$ (Combining (7) and (8))

$$\frac{\partial \phi}{\partial z} - \frac{\omega^2}{g}\phi = 0. \tag{9}$$

Impermeable sea-bed condition

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{at} \quad z = h(x, y). \tag{10}$$