Solutions outline

$Q1.(a)$ x_i = number of nurses starting their week on day-i

$$
\begin{array}{rllllll}
\text{Minimize} & \sum_{i=1}^{7} x_i & & & & \\
\text{Subject to} & x_1 & & +x_4 + x_5 & +x_6 + x_7 & \geq d_1 \\
& x_1 & +x_2 & +x_5 & +x_6 + x_7 & \geq d_2 \\
& x_1 & +x_2 + x_3 & & +x_6 + x_7 & \geq d_3 \\
& x_1 & +x_2 + x_3 & +x_4 & +x_7 & \geq d_4 \\
& x_1 & +x_2 + x_3 & +x_4 + x_5 & & \geq d_5 \\
& & x_2 + x_3 & +x_4 + x_5 & +x_6 & \geq d_6 \\
& & x_3 & +x_4 + x_5 & +x_6 + x_7 & \geq d_7 \\
& x_i \geq 0 & & & &
\end{array}
$$

(b) Idea: $|x| = \max\{-x, x\}$

$$
\begin{array}{rl}
\text{Minimize} & a + b + c \\
\text{Subject to} & x + y \leq 1 \\
& 2x + z = 3 \\
& a \geq x, a \geq -x \\
& b \geq y, b \geq -y \\
& c \geq z, c \geq -z
\end{array}
$$

$Q2.$

$$
\begin{array}{rl}
\text{Minimize} & x_1 + 4x_2 + x_3 \\
\text{Subject to} & 2x_1 - 2x_2 + x_3 = 4 \\
& x_1 - x_3 = 1 \\
& x_2, x_3 \geq 0
\end{array}
$$

$$
\begin{array}{rl}
\text{Minimize} & y_1 - y_2 + 4x_2 + x_3 \\
\text{Subject to} & 2y_1 - 2y_2 - 2x_2 + x_3 = 4 \\
& y_1 - y_2 - x_3 = 1 \\
& y_1, y_2, x_2, x_3 \geq 0
\end{array}
$$

$$\begin{array}{rl} \text{Minimize} & x_1 - x_2 + 4x_3 + x_4 \\ \text{Subject to} & 2x_1 - 2x_2 - 2x_3 + x_4 = 4 \\ & x_1 - x_2 - x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Finding initial BFS

$$\begin{array}{rl} \text{Minimize} & -x_5 - x_6 \\ \text{Subject to} & 2x_1 - 2x_2 - 2x_3 + x_4 + x_6 = 4 \\ & x_1 - x_2 - x_4 + x_5 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

$$\begin{array}{rcc} x_6 = 4 & -2x_1 + 2x_2 & +2x_3 - x_4 \\ x_5 = 1 & -x_1 + x_2 & + x4 \\ \hline z = -5 & +3x_1 - 3x_2 & -2x_3 \end{array}$$

$$\begin{array}{rcc} x_6 = 2 & + 2x_3 & -3x_4 + 2x_5 \\ x_1 = 1 & +x_2 & +x_4 - x5 \\ \hline z = -2 & - 2x_3 & +3x_4 - 3x_5 \end{array}$$

$$\begin{array}{rcc} x_4 = \frac{2}{3} & + \frac{2}{3}x_3 & +\frac{2}{3}x_5 - \frac{1}{3}x_6 \\ x_1 = \frac{5}{3} & +x_2 + \frac{2}{3}x_3 & -\frac{1}{3}x_5 - \frac{1}{3}x_6 \\ \hline z = & & -x_5 - x_6 \end{array}$$

Original Simplex tableau with $B = (1, 4)$

$$\begin{array}{rc} x_4 = \frac{2}{3} & + \frac{2}{3}x_3 \\ x_1 = \frac{5}{3} & +x_2 + \frac{2}{3}x_3 \\ \hline z = \frac{7}{3} & + \frac{16}{3}x_3 \end{array}$$

$[z = x_1 - x_2 + 4x_3 + x_4 = (\frac{5}{3} + x_2 + \frac{2}{3}x_3) - x_2 + 4x_3 + (\frac{2}{3} + \frac{2}{3}x_3) = \frac{7}{3} + \frac{16}{3}x_3]$
$z_{opt} = \frac{7}{3}$

$Q3.(b)$ Primal: $\max c^T x$ Subject to $Ax \leq b$and $x \geq 0$
Dual: $\min b^T y$ Subject to $A^T y \geq c$and $y \geq 0$
Any feasible solution of the following LP gives an optimal solution (using duality theorem)

$$\begin{array}{rl} \text{Maximize} & f(.) \\ \text{Subject to} & Ax \leq b \\ & A^T y \geq c \\ & c^T x \geq b^T y \\ & x \geq 0, y \geq 0 \end{array}$$