

PH-101:Relativity and Quantum Mechanics

Special Theory of Relativity (5 Lectures)

Text Book:1. **An Introduction to Mechanics**

Author: Danieal Kleppner & Robert Kolenkow

2. **Introduction to Special Relativity**

Author: Robert Resnick

This Week: Thursday(Today), Friday(Tomorrow)

Next Week: Monday, Thursday

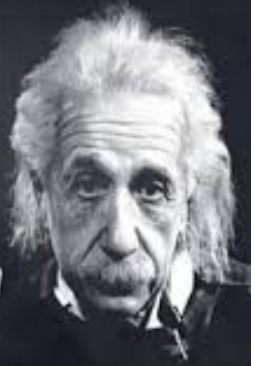
Quantum Mechanics (9 Lectures)

Group-II, IV

$$E = mc^2$$

*The difference between
stupidity and genius
is that genius has its
limits.*

- Albert Einstein



Nagasaki



Hiroshima

Some facts

Einsteinian World

Macroscopic world of our ordinary experiences, the speed **U** of any object is always much less than **C**.

$$C = 3 \times 10^8 \text{ m/s}$$

Example: artificial satellite: $u/c = 0.000027$,

Sound speed: $u/c = 0.000001$

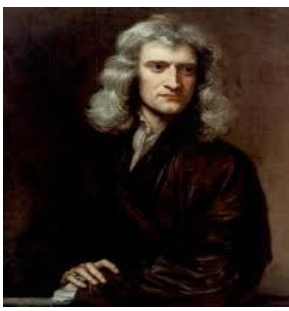
Newtonian World

Microscopic world:

Electron moving in 10 MeV potential difference: $u/c = 0.9988$

Electron moving in 40 MeV potential difference: Newtonian $u/c = 1.9976$

Experimental $u/c = 0.9999$



Newtonian World?

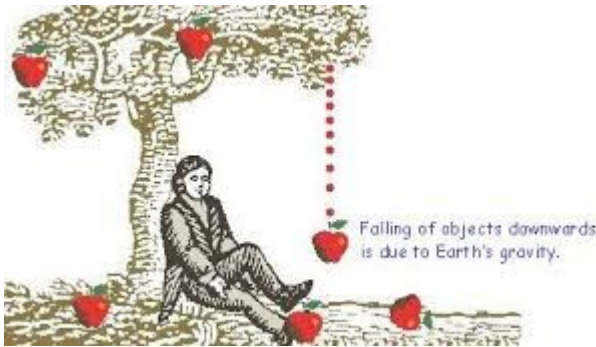
$$m \dot{\mathbf{v}} = \mathbf{f} \quad ; \quad I \dot{\boldsymbol{\omega}} = \mathbf{N}$$

January 4, 1643,

March 31, 1727

Newton's laws of Mechanics in Macroscopic world

You know the Law and applied it in various physical situations



Foundation of Newtonian Mechanic was so robust that nobody has dared to say that this is wrong or has to be modified.

Euler, D'Alembert, Lagrange, Hamilton, Laplace

Generalization of mechanics, Mathematical Structure

Farady, Coulomb, ohm, Maxwell

Electromagnetism

Long time has passed



230 year after

Is Newton's world
valid for any velocity?

Certainly Not!!



March 14, 1879,

April 18, 1955,

100 year after



September, 2016

Lot of open problems

Enough time has passed,

IT COULD BE YOUR WORLD!!

Symmetry Principles

Try to convince you: It is “Einsteinian world”!!

Year 1905: Einstein, motivated by a desire to gain deeper insight into the nature of electromagnetism, push forward the idea of special theory of relativity

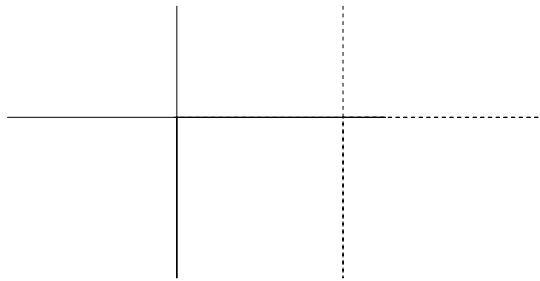
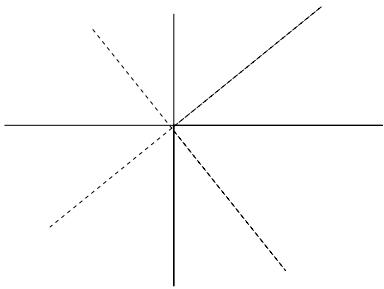


Where is the velocity U

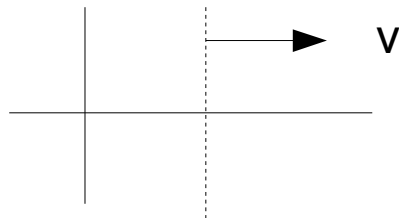
Symmetries

Any “physical law” should be invariant under some **special set of transformations**.

- 1) **Rotation** and **translation** of the given coordinate system



- 2) Transformation from one observer (S) to other (S') who is moving with constant velocity with respect to S.



Linear Transformations

Rotation and Translation

Rotation in x-y plane

Translation along x

Tutorial
problem

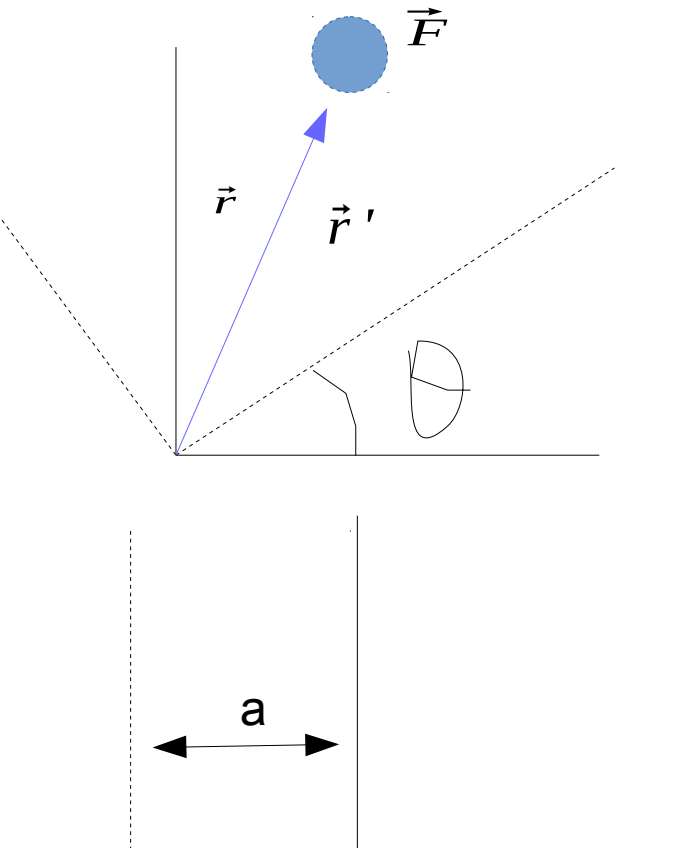
$$\begin{aligned}x' &= x \cos \theta + y \sin \theta \\y' &= -x \sin \theta + y \cos \theta \\z' &= z\end{aligned}$$

$$\begin{aligned}x' &= x + a \\y' &= y \\z' &= z\end{aligned}$$

We do not touch “t” in Galilean transformation

Newton's Law is invariant under space rotation

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}(r) \quad \xrightarrow{\quad} \quad m \frac{d^2 \vec{r}'}{dt^2} = \vec{F}'(r')$$



Linear transformation: **Rotation**

Length is invariant under rotation and translation:

$$ds^2 = dx^2 + dy^2 + dz^2 = dx'^2 + dy'^2 + dz'^2 = ds'^2$$

Galilean Transformation(GT)

“Space” and “Time”

Motion is along x-direction

$$\vec{r}' = \vec{r} - \vec{v} t$$

$$x' = x - vt, \quad y' = y, \quad z' = z$$

We do not touch “t” in Galilean transformation

Newton's Law is invariant under GT

$$\frac{d^2 \vec{r}}{dt^2} = \frac{d^2 \vec{r}'}{dt^2}$$

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}(\vec{r})$$



$$m \frac{d^2 \vec{r}'}{dt^2} = \vec{F}'(\vec{r}')$$

Inertial frames

Velocity addition: $u' = u - v$

Linear transformation: Rotation

Not talking about: Non-linear transformation:

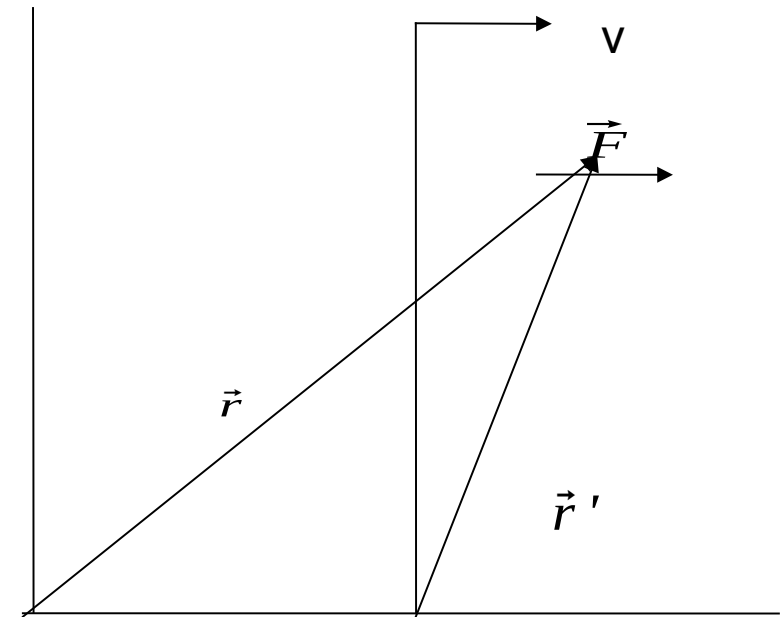


$$\vec{r}' = \vec{r} - \vec{v} t - \frac{1}{2} \vec{f} t^2$$



$$m \frac{d^2 \vec{r}'}{dt^2} = \vec{F}'(\vec{r}') + \vec{f}$$

Depending on f: law changes



No mechanical experiments carried out entirely in one inertial frame can tell the observer what the motion of that frame is with respect to any other Inertial frame

There is no preferred reference frame

We can only speak about relative velocity: Galilean relativity



Electromagnetism and Galilean relativity!!

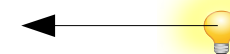


Another Pillar: Maxwell's equation of electromagnetism

June 13, 1831- November 5, 1879,



c: velocity of light



Relative velocity of light: $c' = c - v$

With respect to whom “c” is defined?

Michelson-Morley Experiment: could not prove the existence of any frame

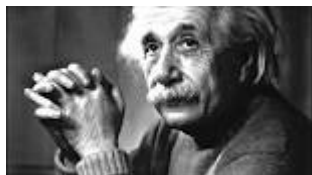
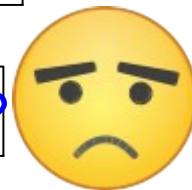
$$\nabla \cdot E = 0, \nabla \cdot B = 0 \quad E : \text{Electric field}$$

$$\nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t} \quad B : \text{Magnetic field}$$

$$\nabla \times E = - \frac{\partial B}{\partial t}$$

Unfortunately these set of equtions are not invariant under GT

Why are we so obsessed with such an “invariance” property under GT?



“Laws” should not be dependent upon “who” is observing!
WHO?

Class of observes: moving with constant velocity relative to each other

Inertial observers

There is a problem!!

1) Electromagnetism?

2) Different relativity principle for Electromagnetism

2) Newtonian Mechanics?

Einstein did series of

1. “thought experiments”: based on light and measurement of time!
2. measurement of time

Einstein was a Genius

worried about basic principle based on which laws are defined

1. Newton's law does not contain “velocity” explicitly
2. Maxwell's equation contains velocity of light “ c ” explicitly

Since velocity of light, C , involves no reference to a medium,

C , must be universal constant!!

This simple but bold statement changes the very notion of “time”

How and why?

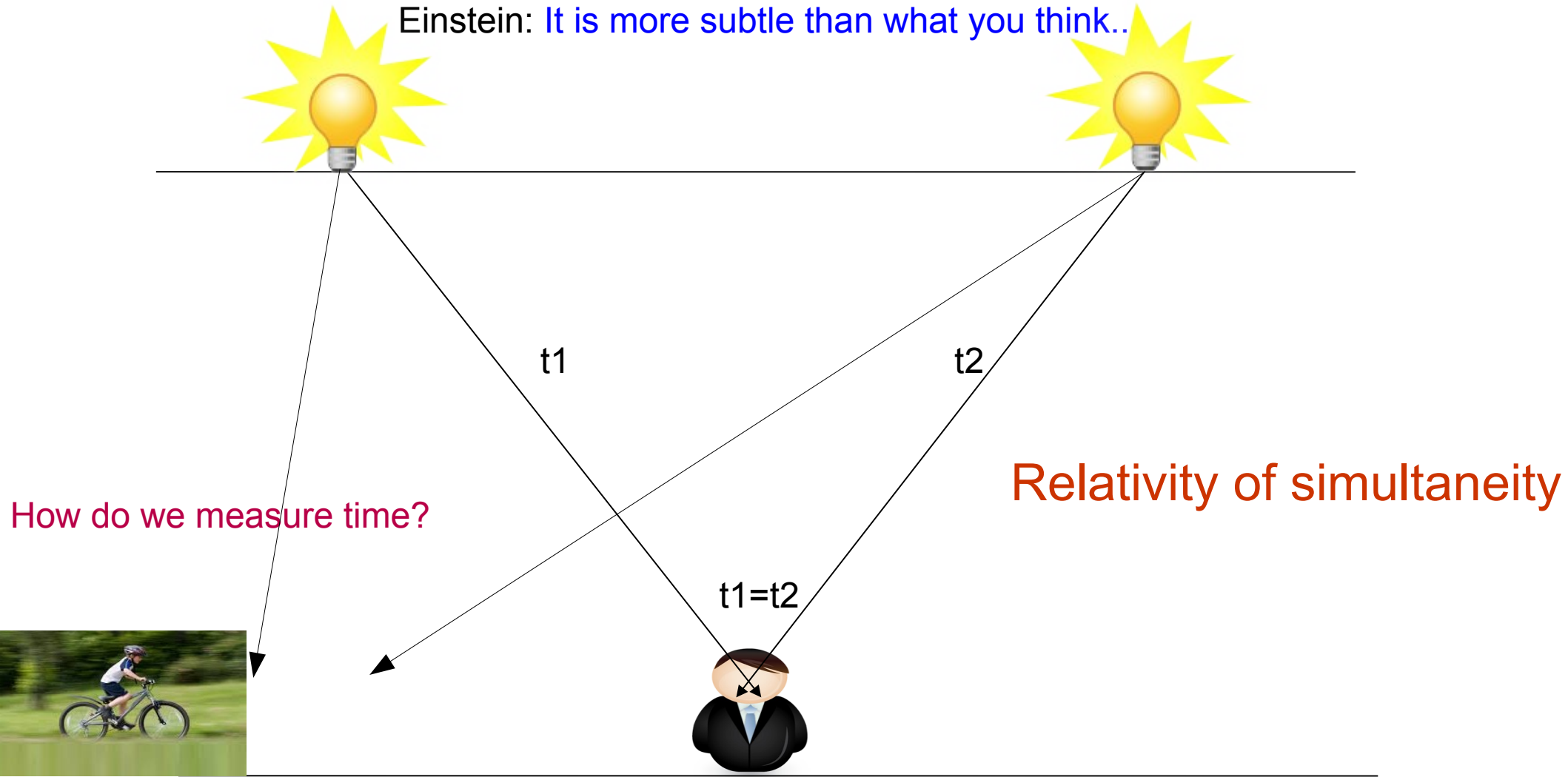
Follow Einstein's path: Do some thought experiments

1. Velocity of light is finite
2. Usual Galileon relativity
3. Every observer has his own clock to measure time
4. Every observer has his own light detector

Newton's law does not talk about how do we measurement the time

Galileo: Time is sacred, it does not transform ...

Einstein: It is more subtle than what you think...



Finite velocity of light plays crucial role in defining observer's "Time"

"Time" is observer dependent !!

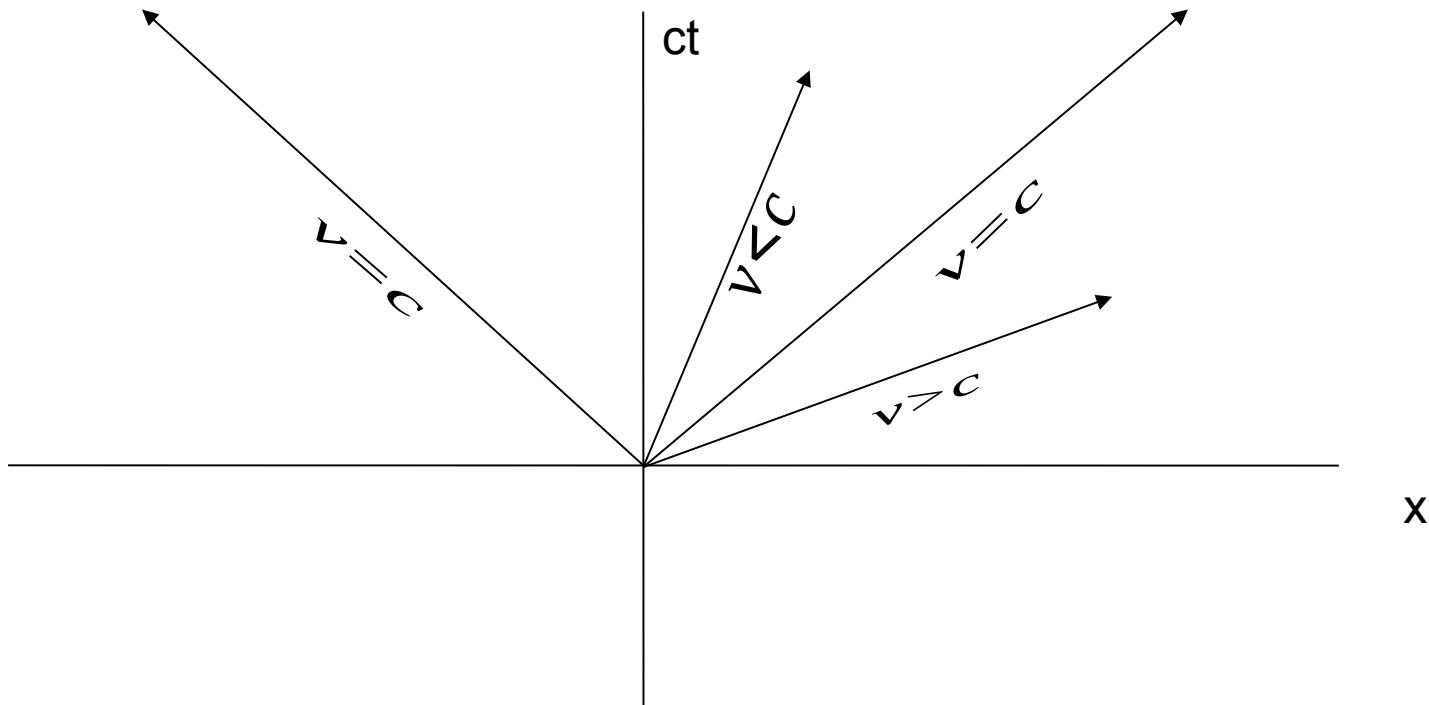
This can be very well explained in space and time diagram

'Space and Time' diagram

Inertial observer: $x = vt$

Galilean relativity does not say anything about the magnitude of "v"

$$x = \left(\frac{v}{c} \right) ct$$



Events are at two different points

Space" and "Time" diagram

Talking about a class of observers:
moving with constant velocity relative
to each other

Inertial Observers

$$x = vt, \quad x = \left(\frac{v}{c} \right) ct$$

$$x' = f(x, t), \quad t' = g(t, x)$$

must be true

event

1

event

2

x

Real Experiment

Events are at two different points

Space" and "Time" diagram

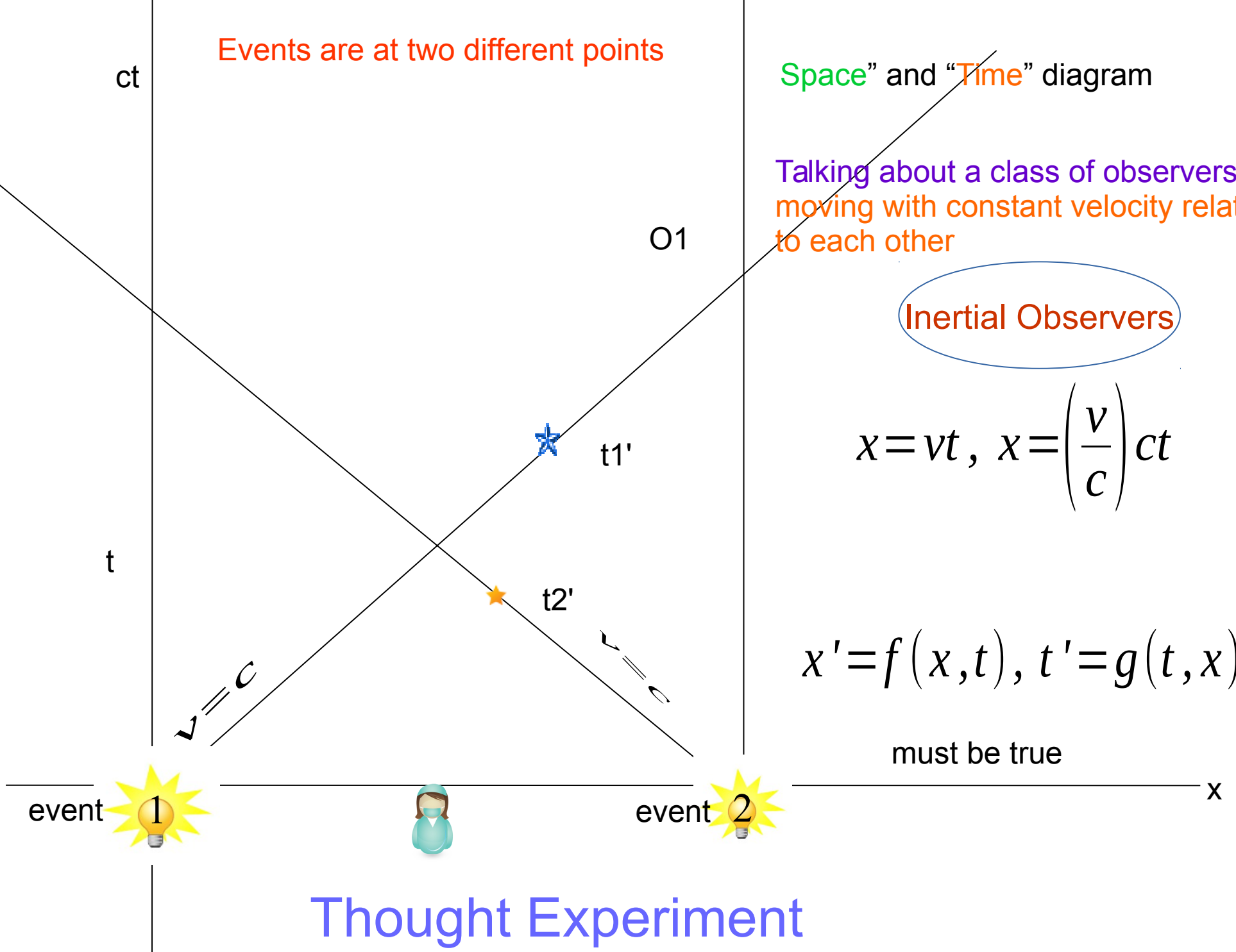
Talking about a class of observers:
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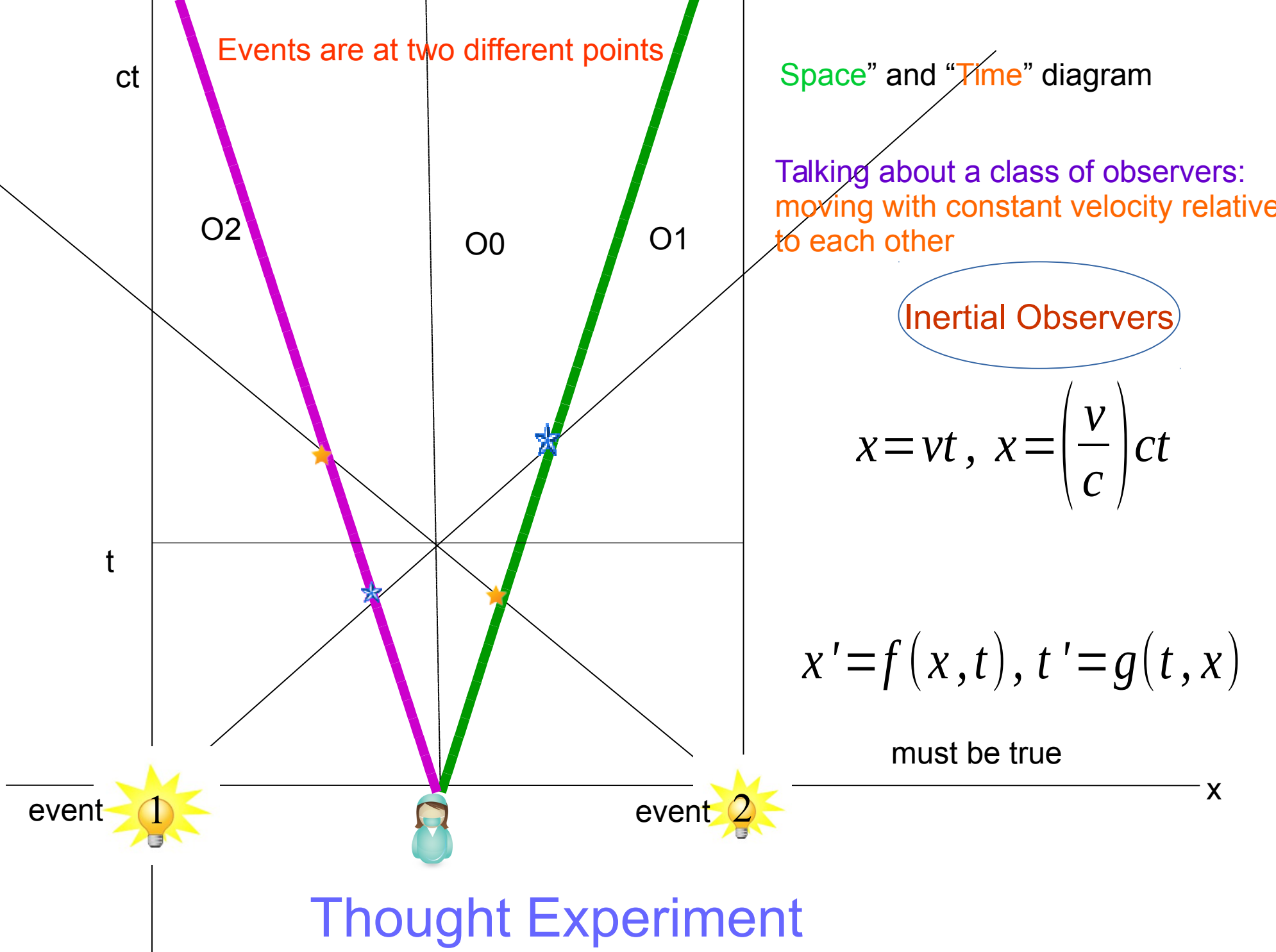
Inertial Observers

$$x = vt, \quad x = \left(\frac{v}{c}\right) ct$$

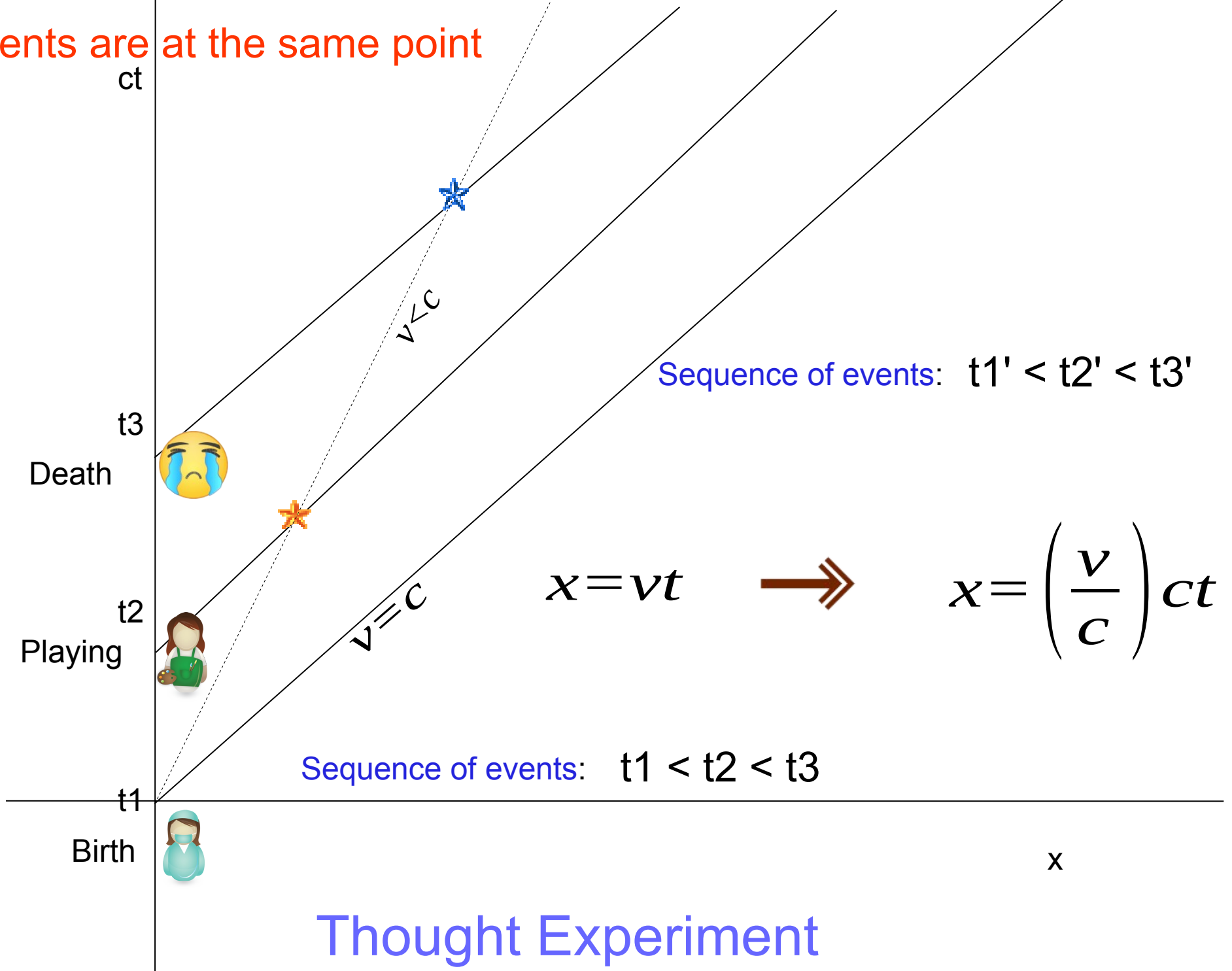
$$x' = f(x, t), \quad t' = g(t, x)$$

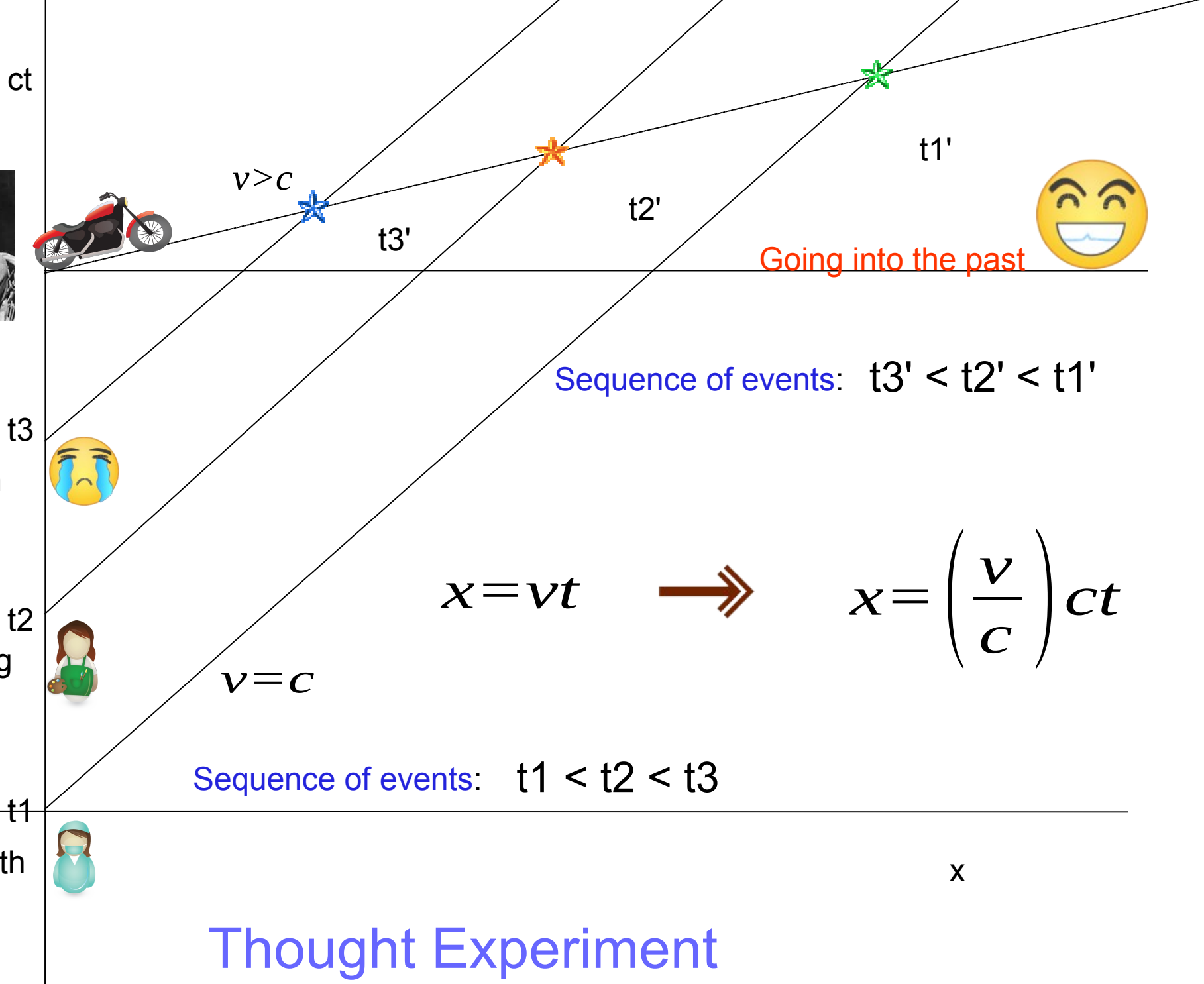
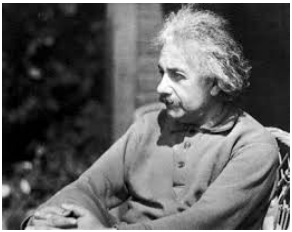
must be true





Events are at the same point





What we have observed so far

1. Measurement of time does depend on the position of an event
2. There should exist a limiting velocity which is the velocity of light
3. Galileo's idea of observer independence should be valid,
However, Galilean transformation has to be modified

Einstein's postulates:

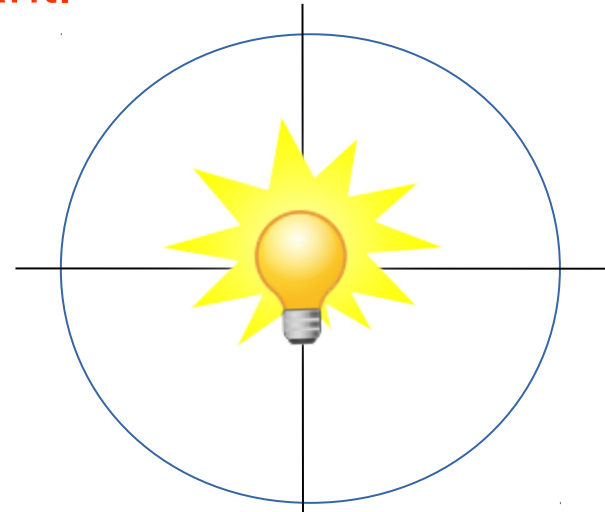
- 1) The laws of physics are the same in all inertial frame.
- 2) The speed of light in free space is same in all inertial frame.

What we have learnt: (x, t) & (x', t') : We must find out a **linear relation** among them **keeping velocity of light constant**.

Assuming:

$$x' = A(x - vt), t' = B(t - f(x))$$

$$y' = y, z' = z$$



$$x^2 + y^2 + z^2 = c^2 t^2$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

$$A^2(x^2 - 2xvt + v^2 t^2) + y^2 + z^2 = B^2 c^2 (t^2 - 2f(x)t + f(x)^2)$$

$$(A^2 x^2 - B^2 c^2 f(x)^2) + 2t(-A^2 xv + B^2 c^2 f(x)) + y^2 + z^2 = (B^2 - A^2 \frac{v^2}{c^2}) c^2 t^2$$

$= x^2$
 $= 0$
 $= 1$

$$f(x) = \frac{A^2 v x}{B^2 c^2}$$

Consistent solution exists for:

$$A = B$$

$$A^2 \left(1 - \frac{v^2}{c^2} \right) = 1$$

Lorentz Transformations

$$x' = A(x - vt), t' = B\left(t - \frac{vx}{c^2}\right)$$

$$y' = y, z' = z$$

$$A = B = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma$$

$$x' = \gamma(x - vt), t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$y' = y, z' = z$$

Newtonian Limit(non-relativistic)

$$\frac{v}{c} \ll 1$$

$$x' = (x - vt), t' = t$$
$$y' = y, z' = z$$

Since all observers are equivalent, the inverse transformation would be:

$$x = \gamma(x' + vt'), t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$y = y', z = z'$$

Time and space get inter-connected

“Space” and “Time” $\Rightarrow$ “Space-time”



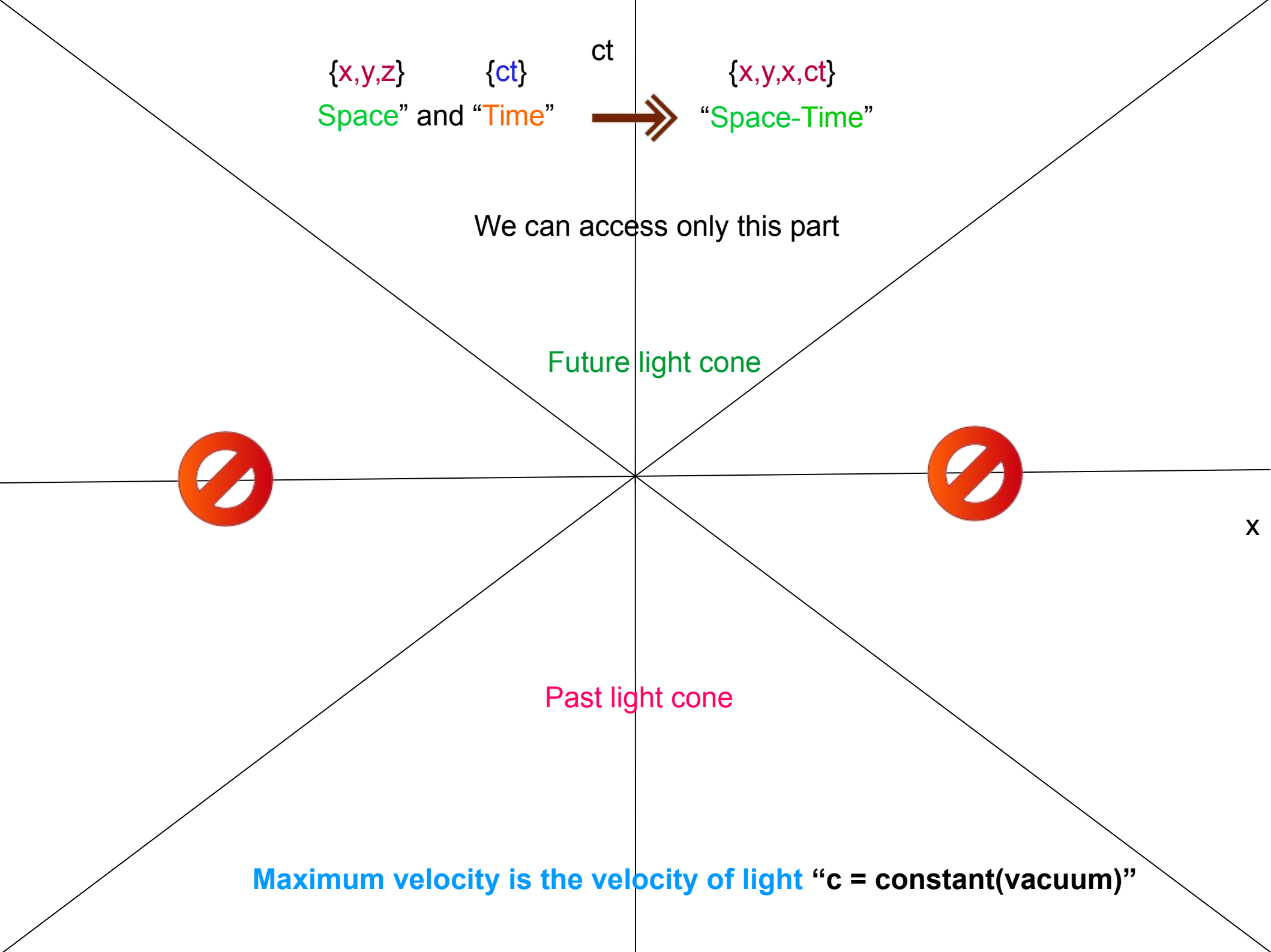
$\{x,y,z\}$ $\{ct\}$ ct
Space and Time $\Rightarrow$ $\{x,y,x,ct\}$
"Space-Time"

We can access only this part

Future light cone

Past light cone

Maximum velocity is the velocity of light "c = constant(vacuum)"



Offsprings

Time dilation (Moving clock runs slow!!)



Length Contraction (Running cat is safe for being shorter!!)

Relativistic Doppler effect



Time Dilation

Two events: **emission** and **absorption** of light at the same point

$$t_a = \gamma \left(t'_a + \frac{v x'}{c^2} \right)$$

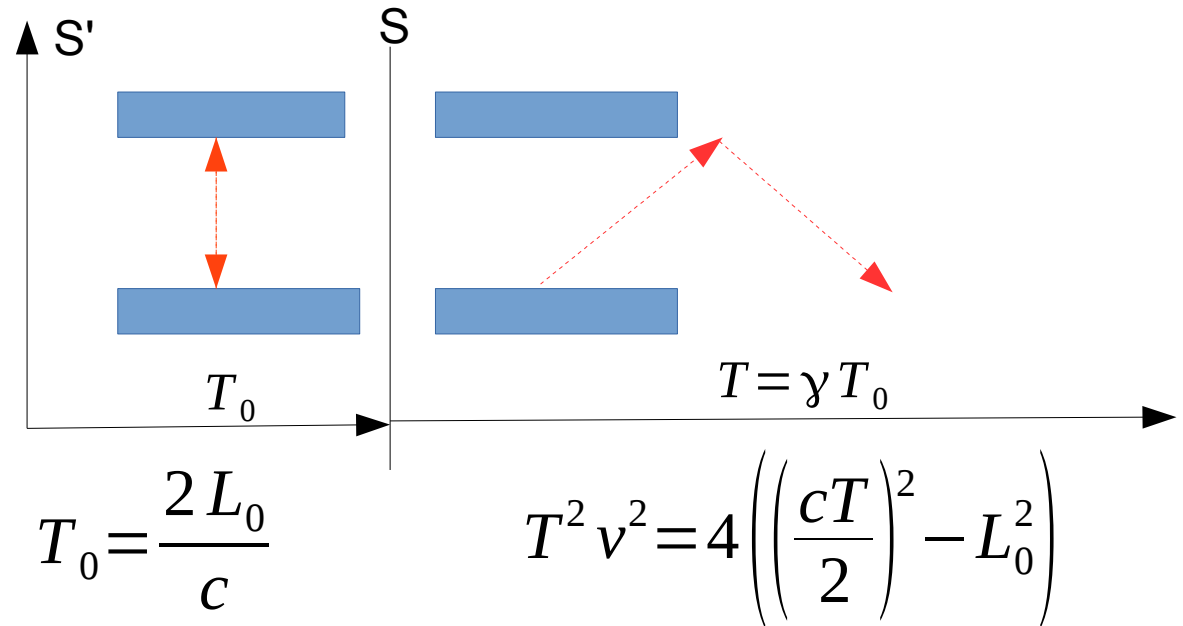
$$t_b = \gamma \left(t'_b + \frac{v x'}{c^2} \right)$$

$$t_b - t_a = \gamma (t'_b - t'_a)$$

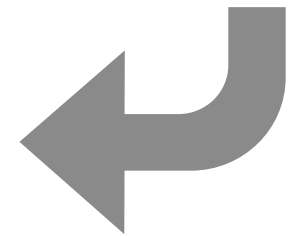
$$T = \gamma T_0$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} > 1$$

$$T > T_0$$



$$T = \frac{2L_0}{c \sqrt{1 - \frac{v^2}{c^2}}} = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$



Moving clock runs slow

Length contraction

Before talking any such counter intuitive things,
let us talk about what do mean by length measurement

Length of a stick is the distance between its ends measured at the **same instant of time**



Length contraction

Condition: Length of a stick is the distance between its ends at the **same instant of time**

$$x'_a = \gamma(x_a - vt)$$

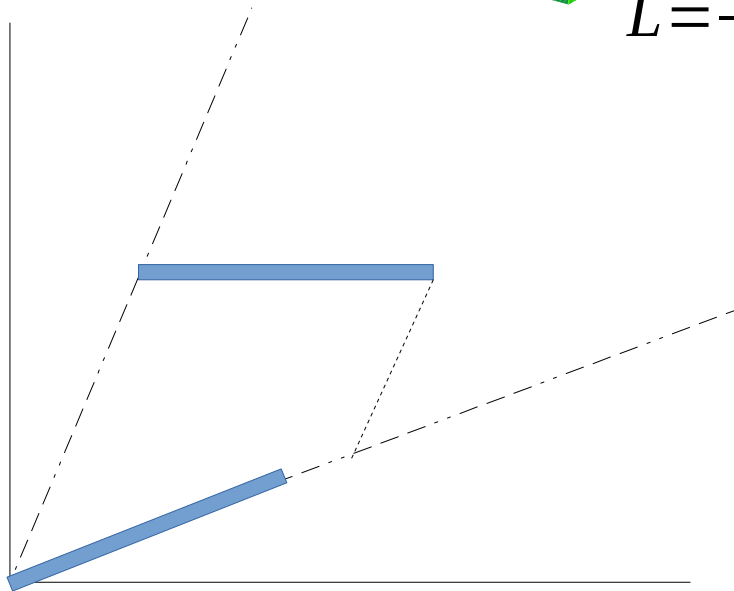
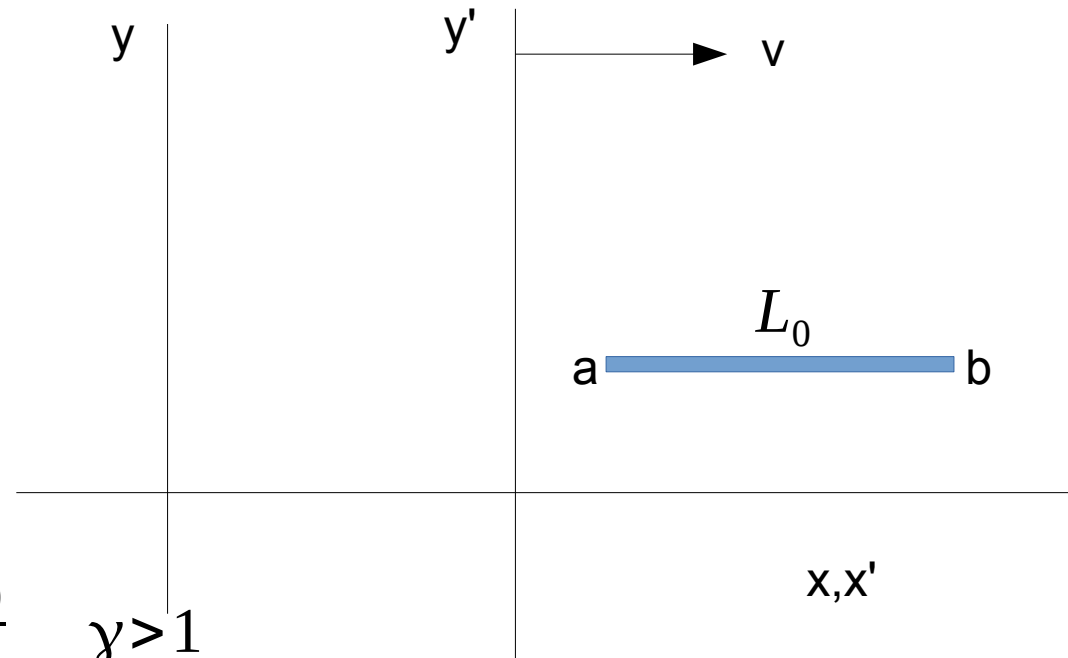
$$x'_b = \gamma(x_b - vt)$$

$$x'_b - x'_a = \gamma(x_b - x_a)$$

$$L_0 = \gamma L$$

$$L = \frac{L_0}{\gamma}$$

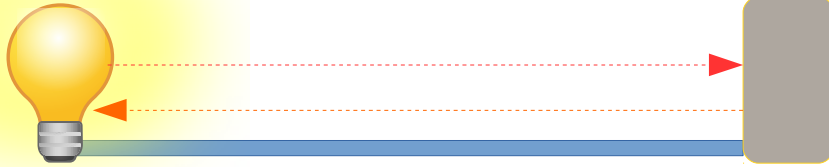
$$\gamma > 1$$



Therefore: $L < L_0$

Two events: **emission** and **absorption** of light at the same point

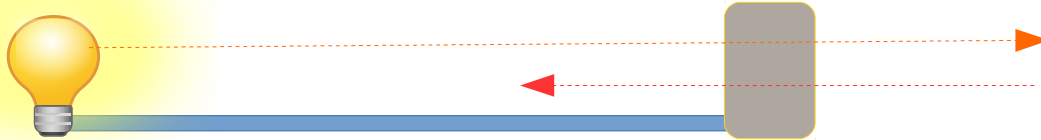
S'-frame



$$T_0 = \frac{2L_0}{c}$$

$$T_0 = t'_2 - t'_1$$

S-frame



$$\Delta t_1 = \frac{L}{c} + \frac{\Delta t_1 v}{c} ; \Delta t_2 = \frac{L}{c} - \frac{\Delta t_2 v}{c}$$

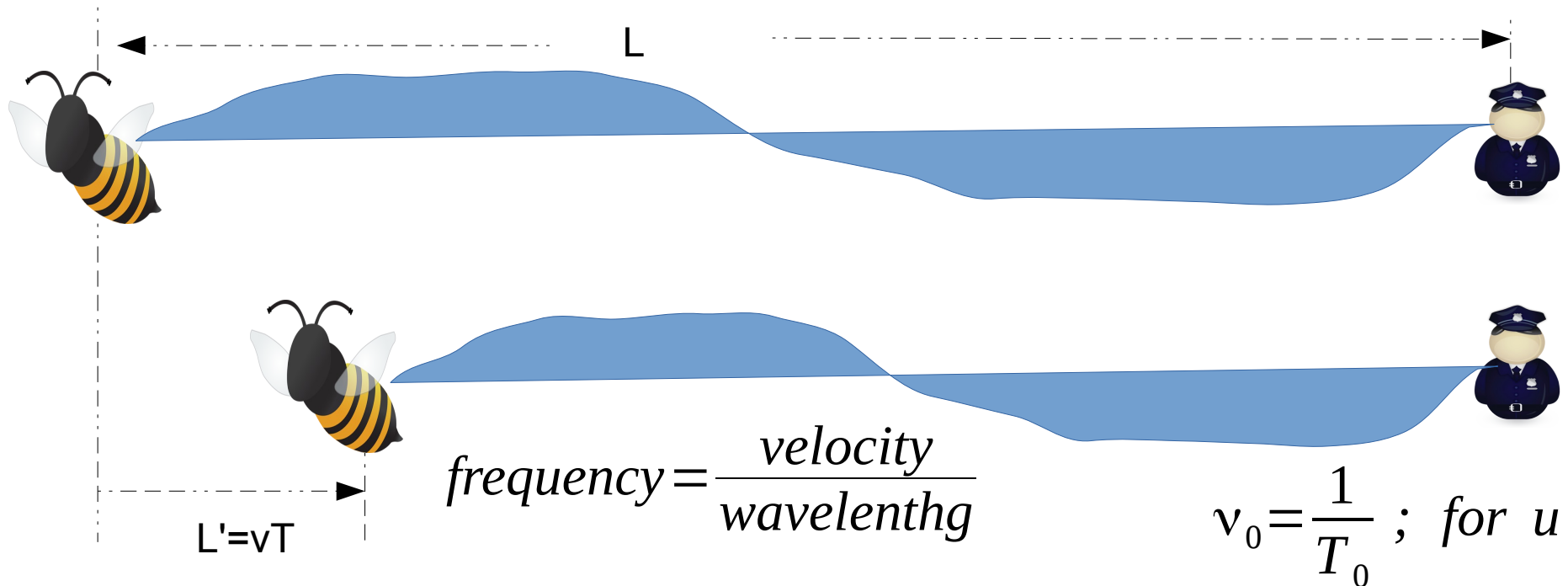
$$T = t_2 - t_1$$

$$T = \Delta t_1 + \Delta t_2 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2Lc}{(c^2 - v^2)}$$

We learnt:

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \frac{2Lc}{(c^2 - v^2)} = \frac{2L_0}{c \sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow L = \left(\sqrt{1 - \frac{v^2}{c^2}} \right) L_0$$

The Doppler effect



$$v_d = \frac{u}{L - L'} = \frac{u}{uT - vT} = \frac{1}{T} \frac{u}{u - v} = \frac{1}{T_0} \frac{u}{u - v}$$

(Newtonian) Non-relativistic

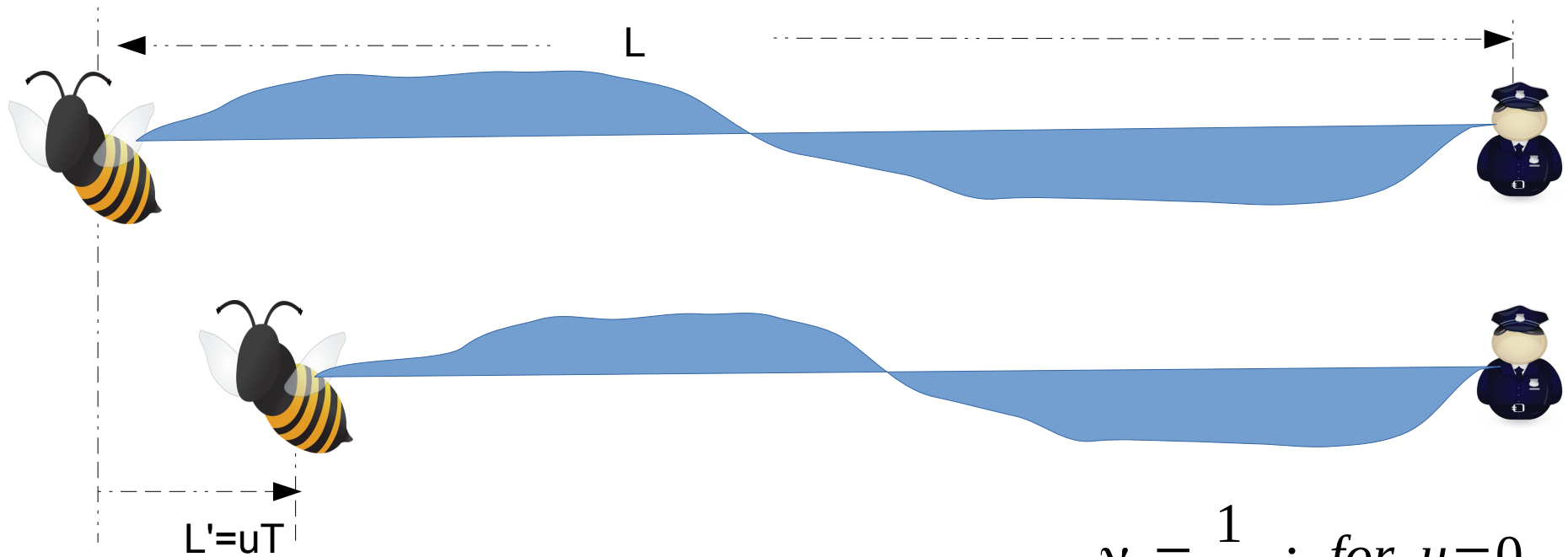
$$T = T_0$$

For light signal: $u = c$

$$v_d = \frac{1}{T} \frac{u}{u - v} = \frac{1}{\gamma T_0} \frac{c}{c - v} = v_0 \sqrt{\frac{c + (+v)}{c + (-v)}}$$

Relativistic: $T = \gamma T_0$

The Doppler effect



$$v_0 = \frac{1}{T_0} ; \text{ for } u=0$$

Non-relativistic: $T = T_0$

$$v_d = \frac{u}{L - L'} = \frac{u}{uT - vT \cos \theta} = \frac{1}{T} \frac{u}{u - v \cos \theta}$$

$$v_d = \frac{1}{\gamma T_0} \frac{c}{c - v \cos \theta}$$

Relativistic: $T = \gamma T_0$

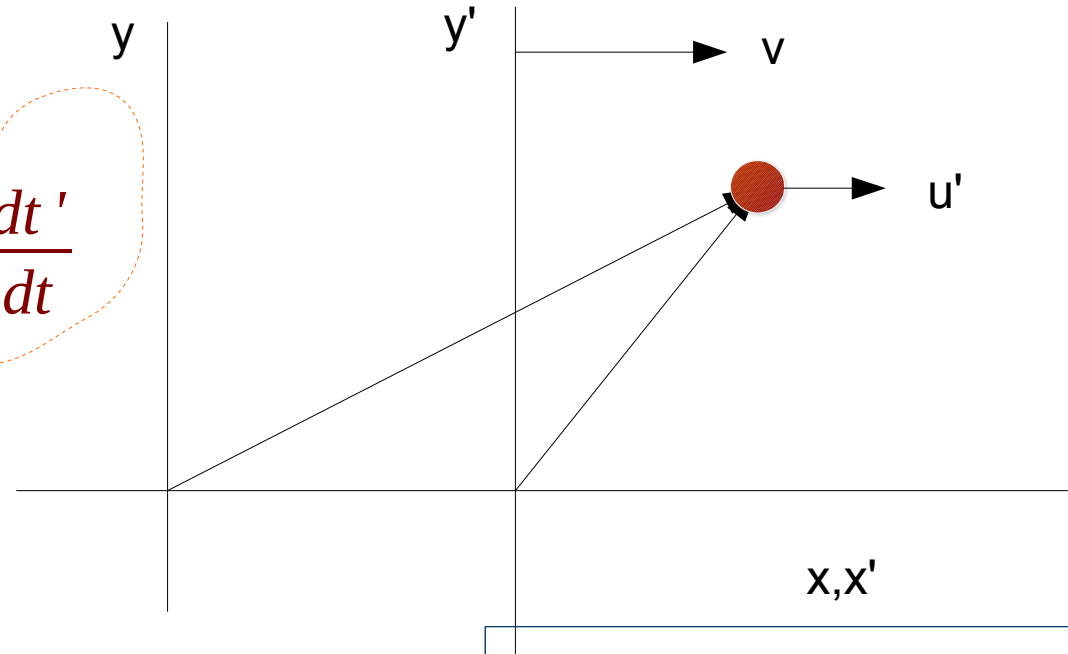
Velocity addition theorem

Assuming: u' is the velocity with respect to the moving frame

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\frac{dx}{dt} = u = \gamma \frac{d(x' + vt')}{dt} = \gamma \frac{d(x' + vt')}{dt'} \frac{dt'}{dt}$$

$$u = \gamma^2(u' + v) \left(1 - \frac{vu}{c^2}\right)$$



$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

$$u_{max} = c$$

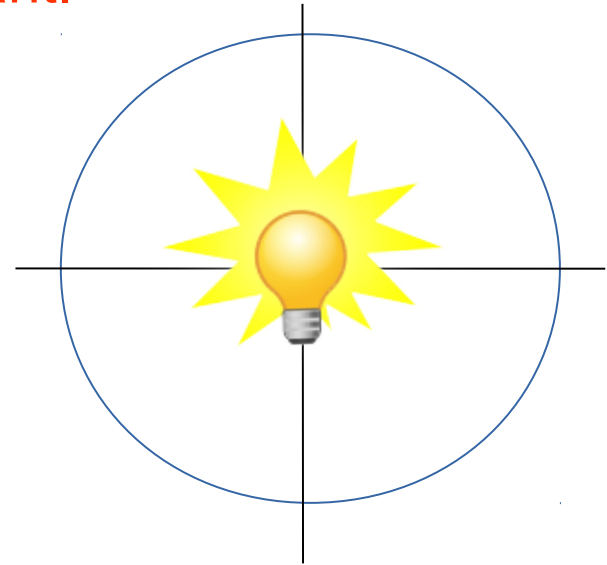
Lorentz Transformation: Invariance property

What we have learnt: (x, t) & (x', t') : We must find out a **linear relation** among them **keeping velocity of light constant**.

Assuming: $x' = A\left(x - vt\right), t' = B\left(t - \frac{vx}{c^2}\right)$

$$y' = y, z' = z$$

$$x^2 + y^2 + z^2 = c^2 t^2$$
$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$



Let us understand how do we derive “A”

If you think little deep, actually we demand:

$$S'^2 = -c^2 t'^2 + x'^2 + y'^2 + z'^2 = -c^2 t^2 + x^2 + y^2 + z^2 = S^2$$

For light, $S = S' = 0$

$$A^2 \left(1 - \frac{v^2}{c^2}\right) = 1$$

Lorentz Transformation: Invariance property

$$x' = \gamma(x - vt), t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$y' = y, z' = z$$

$$S^2 = -c^2 t^2 + x^2 + y^2 + z^2$$

$$S'^2 = -c^2 t'^2 + x'^2 + y'^2 + z'^2$$

$$S^2 = S'^2$$

Transformation is linear in (t, x, y, z)

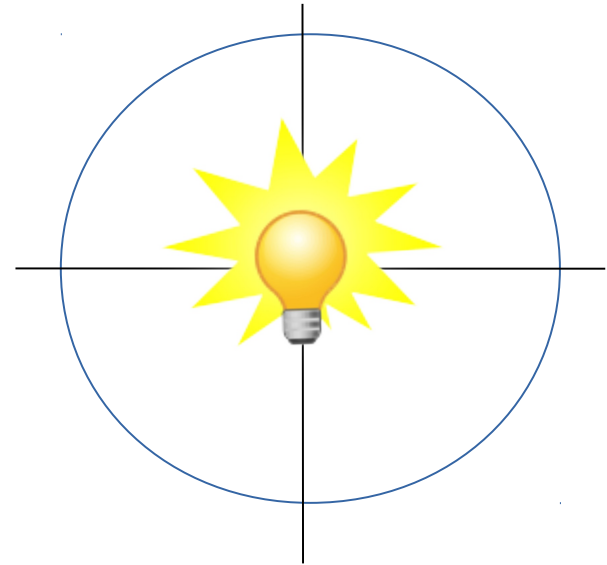
Invariant

$$dx' = \gamma(dx - v dt), dt' = \gamma\left(dt - \frac{v dx}{c^2}\right)$$

$$dy' = dy, dz' = dz$$

Invariant quantities:

$$dS^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = dS'^2$$



$$d\tau^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2} = dt^2 \left[1 - \frac{1}{c^2} \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right) \right]$$



$$= dt^2 \left(1 - \frac{u^2}{c^2} \right)$$

Invariant time: proper-time

Newtonian Rotation+Galileon

$$\begin{aligned} dx' &= dx \cos \theta + dy \sin \theta \\ dy' &= -dx \sin \theta + dy \cos \theta \\ dz' &= dz \end{aligned}$$

$$dx' = dx - v dt, \quad dy' = dy, \quad dz' = dz$$

$$dt' = dt$$

$$ds^2 = dx^2 + dy^2 + dz^2 = ds'^2$$

Distance: measured at same instant of time

Time t : Invariant under all observers

dt

With respect to which we measure
physical variation

Einsteinian Rotation+ Lorentz

$$\begin{aligned} x' &= x \cos \theta + y \sin \theta \\ y' &= -x \sin \theta + y \cos \theta \\ z' &= z \end{aligned}$$

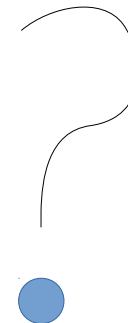
$$dx' = \gamma(dx - v dt), \quad dt' = \gamma\left(dt - \frac{v dx}{c^2}\right)$$

$$dy' = dy, \quad dz' = dz$$

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = ds'^2$$

Proper time τ : Invariant under all observers

$d\tau$



Concept of observer independent time (proper time)

Why do we need to define such a quantity?

Newtonian dynamics: we use time (t) as a **parameter (observer independent)**:
measure all the dynamical **'changes'** of state of a system (position, momentum,...)

Now you know: **Newtonian time is not invariant any more!!**

Define invariant time (proper time)

$$d\tau^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2} = dt^2 \left[1 - \frac{1}{c^2} \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right) \right]$$
$$= dt^2 \left(1 - \frac{u^2}{c^2} \right)$$

$$d\tau = dt \sqrt{1 - \frac{u^2}{c^2}} = dt' \sqrt{1 - \frac{u'^2}{c^2}} = d\tau'$$

u: velocity of the particle under study

Newtonian world

3-dimensional Euclidean space

$$S^2 = x^2 + y^2 + z^2$$

$$dS^2 = dx^2 + dy^2 + dz^2$$

Definitio: three-position

$$r_i \equiv (x, y, z), \quad r_1 = x, \quad r_2 = y, \quad r_3 = z$$

$$dr_i \equiv (dx, dy, dz), \\ dr_1 = dx, \quad dr_2 = dy, \quad dr_3 = dz$$

$$r^2 = \mathbf{r} \cdot \mathbf{r} = r_1^2 + r_2^2 + r_3^2 \\ = x^2 + y^2 + z^2$$

Einstein's world

4-dimensional Einsteinian space-time

$$S^2 = -c^2 t^2 + x^2 + y^2 + z^2$$

$$dS^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

Definitio: Four-position

$$\mathfrak{R}_i \equiv (ict, x, y, z), \\ \mathfrak{R}_1 = ict, \quad \mathfrak{R}_2 = x, \quad \mathfrak{R}_3 = y, \quad \mathfrak{R}_4 = z$$

$$d\mathfrak{R}_i \equiv (ic dt, dx, dy, dz), \\ d\mathfrak{R}_1 = ic dt, \quad d\mathfrak{R}_2 = dx, \quad d\mathfrak{R}_3 = dy, \quad d\mathfrak{R}_4 = dz$$

$$\mathfrak{R}^2 = \mathfrak{R} \cdot \mathfrak{R} = \mathfrak{R}_1^2 + \mathfrak{R}_2^2 + \mathfrak{R}_3^2 + \mathfrak{R}_4^2 \\ = -c^2 t^2 + x^2 + y^2 + z^2$$

Definitions: Position, Momentum, Acceleration

Newtonian

Time (t) as a parameter with respect to which we define all the changes.

Definition: 3-vectors

3-Position:

$$r_i \equiv (x, y, z), r_1 = x \dots$$

3-momentum:

$$p_i = m_0 \frac{dr_i}{dt} \equiv m_0 \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$

3-acceleration:

$$a_i = \frac{d^2 r_i}{dt^2} \equiv \left(\frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2}, \frac{d^2 z}{dt^2} \right)$$

Relativistic

Proper time as a parameter with respect to which we define all the changes.

Definition: 4-vectors

4-Position:

$$\mathfrak{R}_\mu \equiv (ict, x, y, z), \mathfrak{R}_1 = ict \dots$$

4-momentum:

$$\wp_\mu = m_0 \frac{d \mathfrak{R}_\mu}{d\tau} \equiv m_0 \left(ic \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right)$$

4-acceleration:

$$a_\mu = \frac{d^2 \mathfrak{R}_\mu}{d\tau^2} \equiv \left(ic \frac{d^2 t}{d\tau^2}, \frac{d^2 x}{d\tau^2}, \frac{d^2 y}{d\tau^2}, \frac{d^2 z}{d\tau^2} \right)$$

$$\frac{d p_i}{dt} = F_i$$

$$\frac{d \wp_\mu}{d\tau} = f_\mu$$

“dot” product of vectors

What you know

Definitio: **three-vectors**

3-Position: $r_i \equiv (x, y, z) ; r_1 = x \dots$

$$r^2 = \mathbf{r} \cdot \mathbf{r} = \sum_{i=1}^3 r_i r_i = r_1^2 + r_2^2 + r_3^2 \\ = x^2 + y^2 + z^2$$

Any 3-vector (say 3-momentum):

$$p_i = m_0 \frac{dr_i}{dt} \equiv m_0 \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) \\ p^2 = \mathbf{p} \cdot \mathbf{p} = \sum_{i=1}^3 p_i p_i = p_1^2 + p_2^2 + p_3^2 \\ = m_0^2 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\mathbf{A} \equiv A_i \equiv (A_1, A_2, A_3)$$

In a same way

Definitio: **four-vectors**

4-Position: $\mathfrak{R}_\mu \equiv (i c t, x, y, z), r_1 = i c t \dots$

$$\mathfrak{R}^2 = \mathfrak{R} \cdot \mathfrak{R} = \sum_{\mu=1}^4 \mathfrak{R}_\mu \mathfrak{R}_\mu = \mathfrak{R}_1^2 + \mathfrak{R}_2^2 + \mathfrak{R}_3^2 + \mathfrak{R}_4^2 \\ = -c^2 t^2 + x^2 + y^2 + z^2$$

Any 4-vecto (say 4-momentum):

$$\wp_\mu = m_0 \frac{d \mathfrak{R}_\mu}{d\tau} \equiv m_0 \left(i c \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right) \\ \wp^2 = \wp \cdot \wp = \sum_{i=1}^4 \wp_i \wp_i = \wp_1^2 + \wp_2^2 + \wp_3^2 + \wp_4^2 \\ = m_0^2 (-c^2 t'^2 + x'^2 + y'^2 + z'^2)$$

$$A^\mu \equiv (i A_1, A_2, A_3, A_4)$$

Let us look at term by term of 4-momentum

Momentum 4-vector (say 4-momentum): $\wp_{\mu} = m_0 \frac{dr_{\mu}}{d\tau} \equiv m_0 \left(ic \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right)$

$$d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} = d\tau'$$

$$\wp_1 = i m_0 c \frac{dt}{d\tau} = i \frac{m_0 c}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(\frac{i}{c} \right) m c^2$$

$$= i p_t$$

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\wp_2 = m_0 \frac{dx}{d\tau} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{dx}{dt} = m v_x = p_x$$

$$\wp_3 = p_y ; \wp_4 = p_z$$

Relativistic energy

$$\wp_{\mu} = (i p_t, p_x, p_y, p_z) ; \wp^2 = \sum_{i=1}^4 \wp_i \wp_i = -m^2 c^2 + p_x^2 + p_y^2 + p_z^2$$

Let us closely look at the first component of momentum

Momentum 4-vector (say 4-momentum): $\wp_\mu = m_0 \frac{dr_\mu}{d\tau} \equiv m_0 \left(ic \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right)$

$$d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} = d\tau'$$

$$\wp_1 = i m_0 c \frac{dt}{d\tau} = i \frac{m_0 c}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(\frac{i}{c} \right) m c^2 = \frac{i}{c} \left(m_0 c^2 + \frac{1}{2} m_0 v^2 \dots \right) \text{ (for } v \ll c \text{)}$$

KE : Kinetic energy

$$\delta \wp_1 = \wp_1(v) - \wp_1(0) = \left(\frac{i}{c} \right) \frac{1}{2} m_0 v^2 = \left(\frac{i}{c} \right) (KE)$$

Rest mass energy

Einstein's Total **relativistic** energy = **E** = $\left(m_0 c^2 + \frac{1}{2} m_0 v^2 \dots \right) = m c^2$

$$\wp_\mu = \left(i \frac{E}{c}, p_x, p_y, p_z \right) ; \wp^2 = \sum_{i=1}^4 \wp_i \wp_i = - \left(\frac{E^2}{c^2} \right) + p_x^2 + p_y^2 + p_z^2$$

$$\textit{Energy} \equiv \textit{Mass}$$

$$\wp_{\mu} = \left(i \frac{E}{c}, p_x, p_y, p_z \right) \equiv \left(i \frac{E}{c}, \mathbf{p} \right)$$

$$d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} = d\tau'$$

Relativistic energy

$$E = m c^2$$

Relativistic Mass

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\wp^2 = \sum_{i=1}^4 \wp_i \wp_i = - \left(\frac{E^2}{c^2} \right) + p_x^2 + p_y^2 + p_z^2 = -m^2 c^2 + p_x^2 + p_y^2 + p_z^2$$

Relativistic energy of a particle of momentum $\mathbf{p}$, and mass m_0

$$\begin{aligned}
 \wp^2 &= -\left(\frac{E^2}{c^2}\right) + p_x^2 + p_y^2 + p_z^2 = -m^2 c^2 + p_x^2 + p_y^2 + p_z^2 \\
 &= m^2(-c^2 + v_x^2 + v_y^2 + v_z^2) \\
 &= -\frac{m_0^2}{1 - \frac{v^2}{c^2}}(c^2 - v_x^2 - v_y^2 - v_z^2) \\
 &= -m_0^2 c^2
 \end{aligned}$$

4-momentum
Massive particle:

$$\wp^2 = -m_0^2 c^2$$



$$E^2 = p^2 c^2 + m_0^2 c^4$$

Relativistic energy of a massless particle: Photon

$$m_0 = 0$$



$$\wp^2 = 0$$



$$E = p c$$

$$\wp_\mu = \left(i \frac{E}{c}, \mathbf{p}\right) = (i p, \mathbf{p}) = (i |\mathbf{p}|, \mathbf{p})$$

NO Newtonian analog

Conservation Law

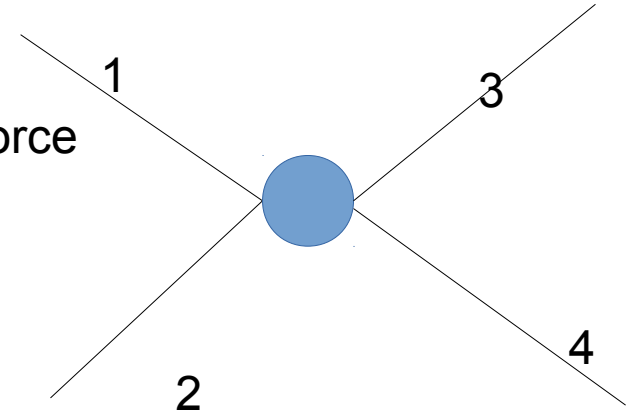
$$\frac{d\wp_{\mu}}{d\tau} = f_{\mu} = 0$$

If there is no external force

$$\wp_{\mu}^1 + \wp_{\mu}^2 = \wp_{\mu}^3 + \wp_{\mu}^4$$

$$E^1 + E^2 = E^3 + E^4$$

$$m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2 = m_3 \mathbf{u}_3 + m_4 \mathbf{u}_4$$



PH-101:Quantum Mechanics

Quantum Mechanics (9 Lectures)

Text: R. Eisberg and R. Resnick, Quantum Physics of
Atoms,Molecules,Solids,Nuclei and Particles, 2nd Ed

Ref: 1.Feynman Lectures, Volume 3

2. Quantum Physics, Stephen Gasiorowicz

$$\lambda = \frac{h}{p}$$

$$H \psi = E \psi \quad \Delta x \Delta p \geq \hbar$$

Quantum Mechanics(lighter note)

More will you be confused, more you will understand

OR

More you understand, more you will be confused

Confusion will be gone: quantify this “confusion” ψ



State of a Coin

$$\psi = \frac{1}{\sqrt{2}} (\text{$$

We will come back to it over and over again, when ever necessary

WHY QUANTUM MECHANICS?

Do not know the answer, no-body knows
This may not be a physics question

What is quantum mechanics?

When we first learnt Newtonian mechanics, we did not bother about Such question.

1. We started with Newton's three laws and then over the years daily life experiences and infinite number of applications made you believe it.
2. We can go ahead with the same logic, start with some set of axioms(rules) and apply it in the microscopic world, and see if it works or not.
3. However, the main problem will be to state those axioms, and do the rest. The reason is all those axioms will sound like non-sence at the beginning.

This is why Newton/Einstein was so popular than Heisenberg/de-Broglie/Schrodinger

They never became a “culture” or a “fashion”, and never will be

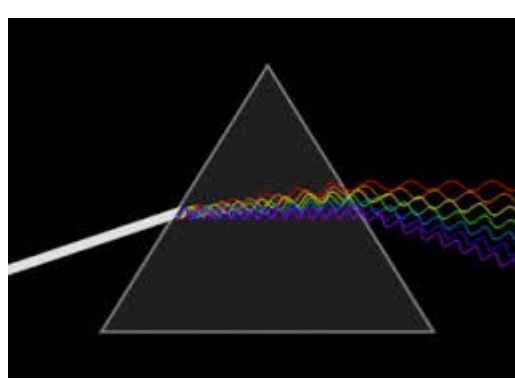
Becasue: It is a theory of probability (Confusion)

There is another world out there, called Quantum world

1. It is a small scale world
2. If everything becomes small, they become delicate, touchy, Sensitive to the “OBSERVATION”
3. If you want to fill them, they will change themselves.

How can we describe such a sensitive world ?

Mathematical model called Quantum Mechanics



LIGHT

Dual nature of light

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

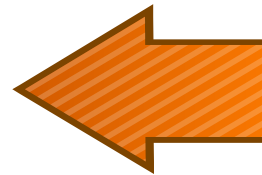
(E,p): Finite

$$E^2 = p^2 c^2 + m_0^2 c^4 = p^2 c^2$$

$$v \rightarrow c$$

$$m_0 \rightarrow 0$$

$$\lambda = \frac{h}{p}$$



Quantum hypothesis

$$E = pc = h \frac{c}{\lambda}$$

Light wave:

$$\psi = A e^{2\pi i (kx - \omega t)}$$

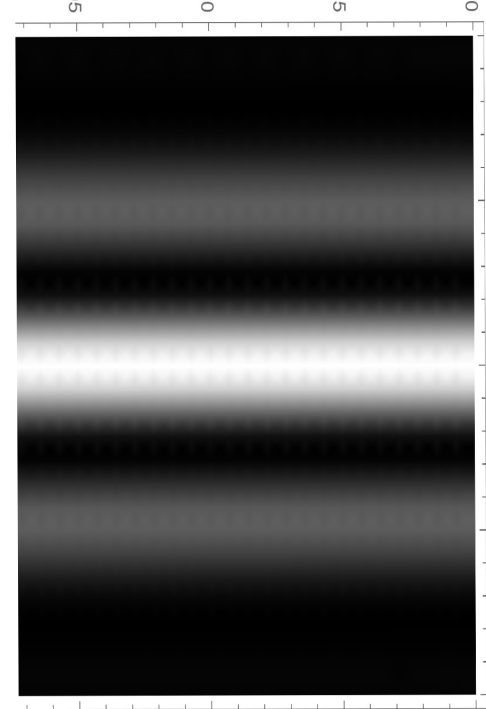
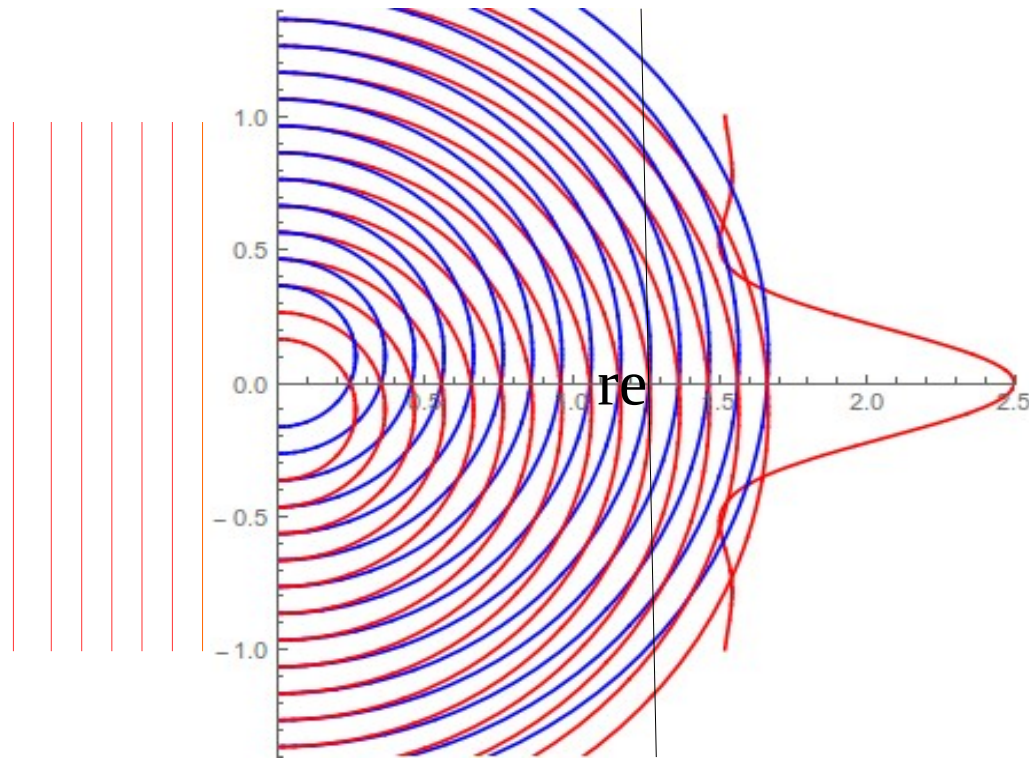
Amplitude: $|\psi|^2$

Probabilty

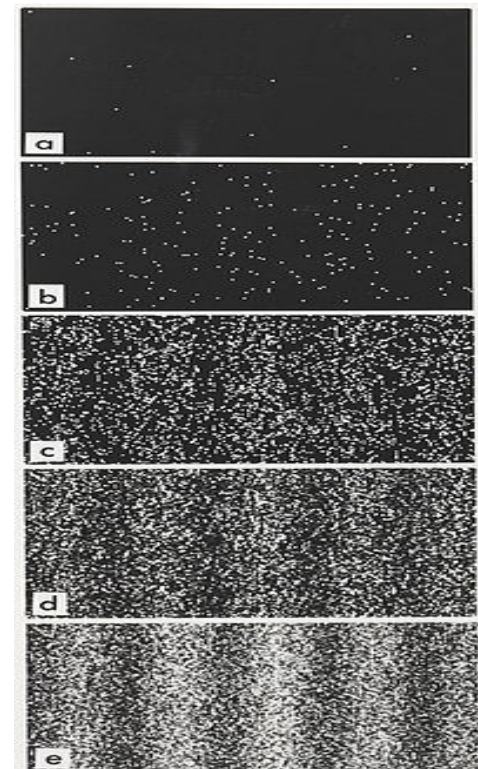


April 23, 1858, - October 4, 1947

LIGHT
WAVE



Light wave: collection of large
number of photons



Reducing the intensity of light = Reducing the number
of photon, not the energy of the photon

Particle

1. Localized object
2. Position, momentum
3. Two particle collide
4. Can not be sub-devided

Wave

1. Extended object
2. Wavelength, Intensity
3. Two waves superpose
4. Can be sub-devided, called wave function

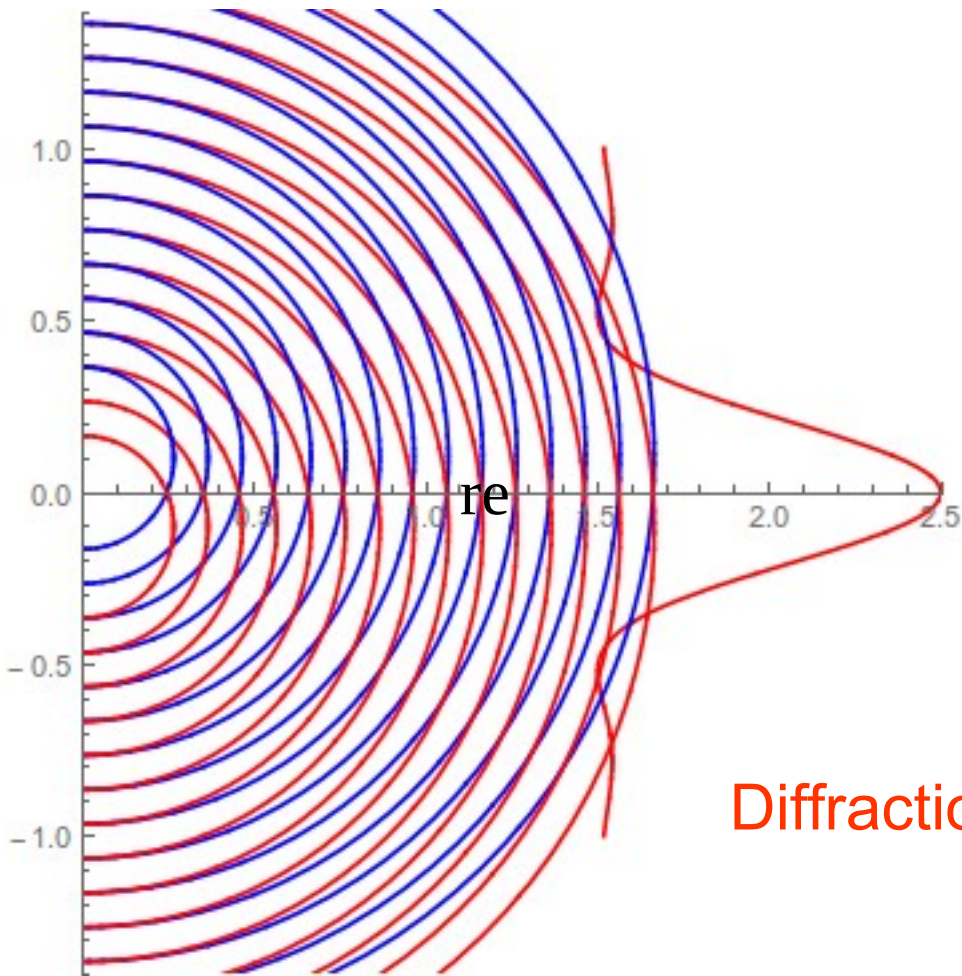
Linear superposition of waves

$$\varphi = \varphi_1 + \varphi_2$$

Total intensity

$$|\varphi|^2 = |\varphi_1|^2 + |\varphi_2|^2 + 2 \Re(\varphi_1 \varphi_2^*)$$

Diffraction pattern: $d \sin \theta = n \lambda$



For single photon !!

Something similar ?

$$\psi = \frac{1}{\sqrt{2}} \left(\text{Image of 1 Rupee coin} + \text{Image of 1 Rupee coin} \right)$$

Wave-particle duality

Single photon: Particle like object

$$E = h \nu \quad ; \quad \lambda = \frac{h}{p}$$

A "wave-function" is assigned to each photon

$$\psi = \psi_1 + \psi_2$$

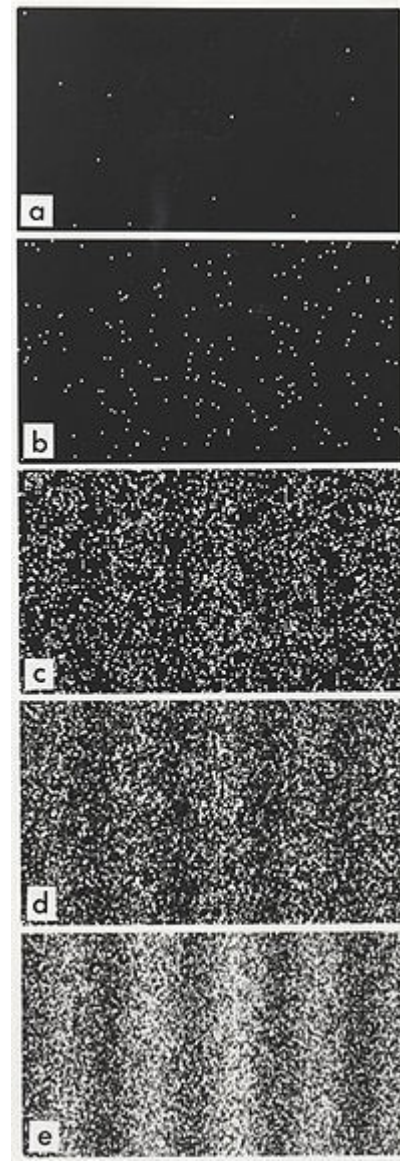
Linear super-position principle

Total probability of
receiving photon

$$|\psi|^2 = |\psi_1|^2 + |\psi_2|^2 + 2 \Re(\psi_1 \psi_2^*)$$

$$\psi_1 = A e^{2\pi i \left(\frac{x}{\lambda} - \nu t \right)}$$

Nature: Probabilistic



Particle-wave Duality (1924)

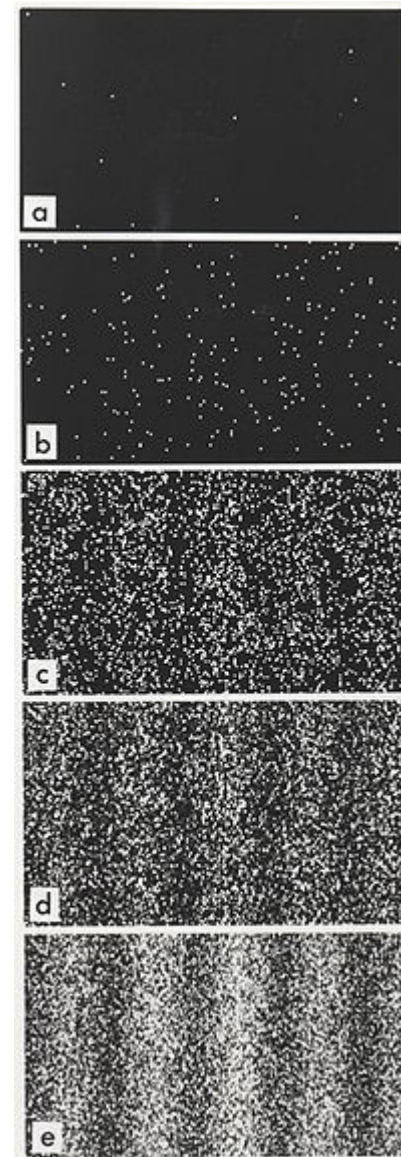
de-Broglie hypothesis

True for any particle of momentum p

$$\lambda = \frac{h}{p}$$

$$E = h \omega$$

Electron diffraction
Experiment



$$v_g = \frac{d\omega}{dk} = v$$

Matter wave:

In general A could be complex number

$$\psi = A e^{i 2 \pi (k x - \omega t)} = A e^{\frac{i}{\hbar} (p x - E t)}$$

$$\psi_{coin} = \frac{1}{\sqrt{2}} (\text{Image of Indian Rupee coin} + \text{Image of Indian Rupee coin})$$

Linear super-position principle of wave function

Born Interpretation

$$\psi(x, t) = \psi_1(x, t) + \psi_2(x, t)$$

Probability amplitude



Probability of finding a particle at x: $P(x, t) = |\psi(x, t)|^2$

Probability of finding a particle
at x and x+dx

$$P(x, t) dx = |\psi(x, t)|^2 dx$$

Group and phase velocity: Concept of uncertainty principle

Simple example:

$$\psi = e^{\frac{i}{\hbar}(p x - E t)}$$

$$P(x, t) = |\psi(x, t)|^2 = 1$$

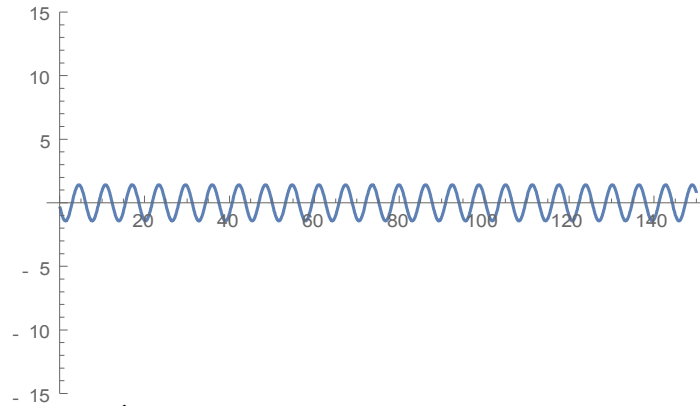
Free particle with
Well defined
momentum

$$\Delta p = 0$$

$$\Delta x = ?$$

Single wave with well define
momentum

$$v_p = \frac{E}{p}$$

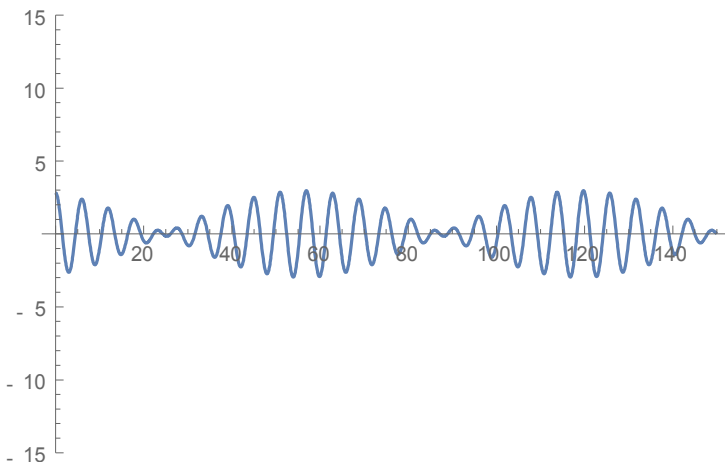


$$\psi = a e^{\frac{i}{\hbar}(p x - E t)} + (a + \delta a) e^{\frac{i}{\hbar}((p + \delta p)x - (E + \delta E)t)} ; P(x, t) = |\psi(x, t)|^2$$

$$\Delta p = \delta p$$

$$\Delta x \approx \frac{2\hbar}{\delta p}$$

$$\Delta p \Delta x \approx 2\hbar$$



$$v_g = \frac{dE}{dp}$$

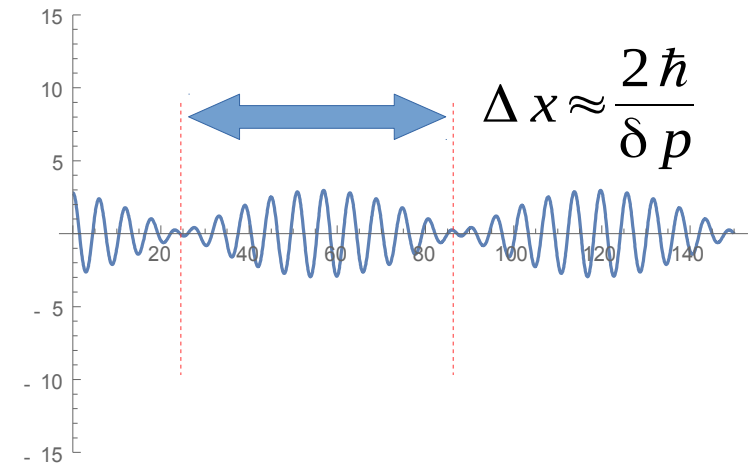
Super-position of two waves with
two different momentum

Matter and wave like behaviour

$$\psi = a e^{\frac{i}{\hbar}(p x - E t)} + (a + \delta a) e^{\frac{i}{\hbar}((p + \delta p)x - (E + \delta E)t)} ; P(x, t) = |\psi(x, t)|^2$$

Consider Real part

$$\begin{aligned} \psi &\approx \cos\left(\frac{p x - E t}{\hbar}\right) + \cos\left(\frac{(p + \delta p)x - (E + \delta E)t}{\hbar}\right) \\ &= 2 \cos\left(\frac{\delta p x - \delta E t}{2\hbar}\right) \cos\left(\frac{(2p + \delta p)x - (2E + \delta E)t}{2\hbar}\right) \end{aligned}$$



$$E = \frac{1}{2} m v^2 ; p = m v$$

Phase velocity

$$v_p = \frac{2E + \delta E}{2p + \delta p} \approx \frac{E}{p} = \frac{v}{2}$$

Group velocity

$$v_g = \frac{\delta E}{\delta p} = \frac{dE}{dp} = v$$

Heisenberg Uncertainty Principle (1927)



Werner Karl Heisenberg
5 Dec 1901-1 Feb 1976

$$\psi(x, t) = a e^{\frac{i}{\hbar}(px - Et)} + (a + \delta a) e^{\frac{i}{\hbar}((p + \delta p)x - (E + \delta E)t)} + \dots$$

$$\psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int d^3p \, \varphi(p) e^{\frac{i}{\hbar}(px - Et)}$$

More and more uncertain the momentum is, more and more localized is the particle,

$$P(x, t) = |\psi(x, t)|^2$$

Reverse must be true

This is a huge conceptual jump from classical to quantum world

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Position and momentum are not the simultaneous Observables !!



Why there should be any uncertainty

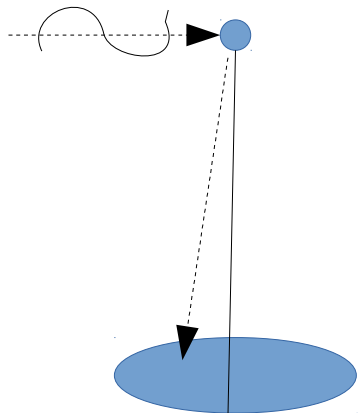
It is the probabilistic nature of the state of a system which leads to the uncertainty

In any probabilistic theory, measurement always disturbs the system in certain way

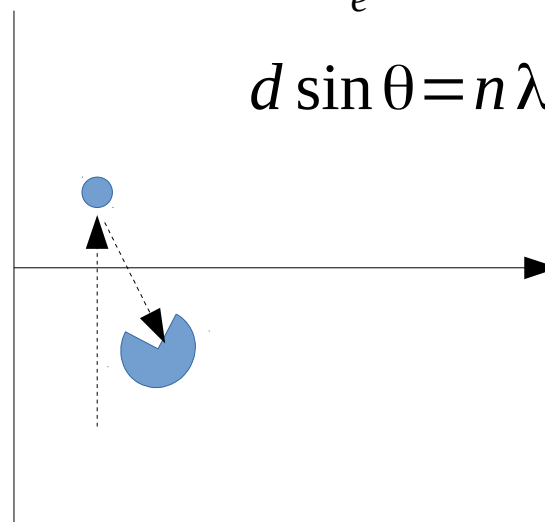
Every observable and its conjugate observable satisfy uncertainty relation

EXAMPLE: position and momentum

$$\Delta x_e \approx \lambda_p, \Delta p_e \approx \frac{h}{\lambda_p}$$



Double slit



$$p_e = \frac{h}{\lambda_e}$$

$$d \sin \theta = n \lambda$$

$$\theta = \frac{\lambda_e}{d} > \frac{\Delta p_e}{p_e}$$

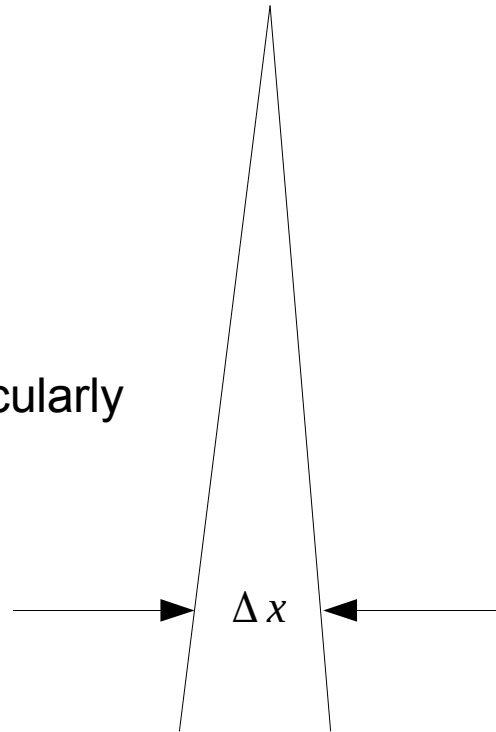
$$\lambda_p > d \quad !!!!$$

Impossible to resolve the slit width



Experiment on Uncertainty

Light is passing
through the slit perpendicularly

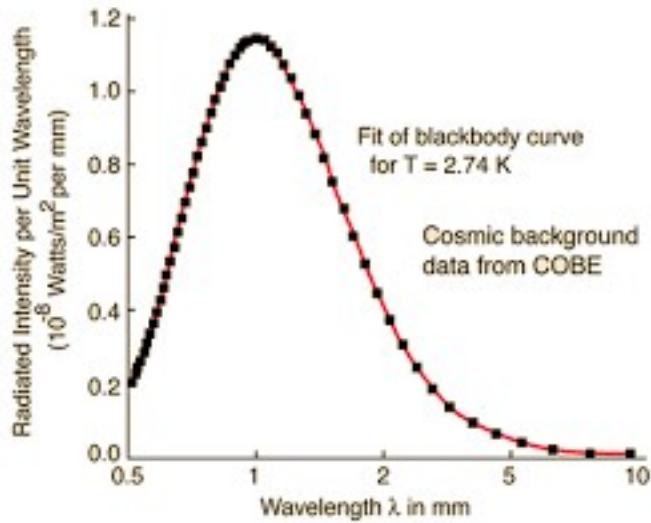


$$\Delta p_x \geq \frac{\hbar}{2 \Delta x}$$

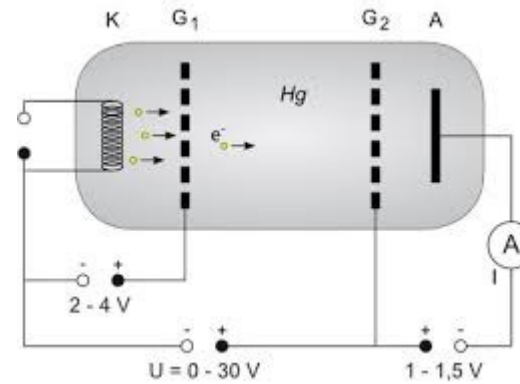
As the source goes upward, the intensity of light also goes down. Therefore, the number of photons is decreasing.

Light will spread along the x-axis as we start to decrease the slit width

Black body spectrum (Max Planck, in 1901)



Franck-Hertz experiment (1914)



Stern-Gerlach experiment (1922)

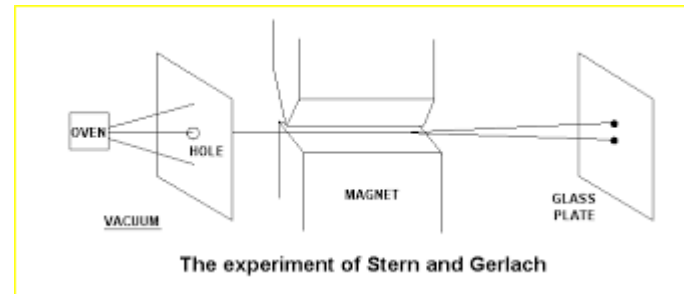
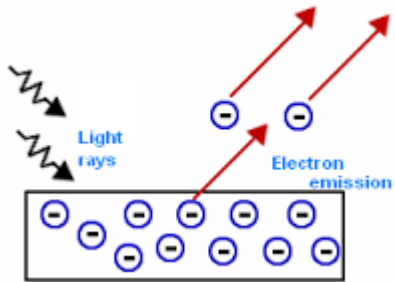
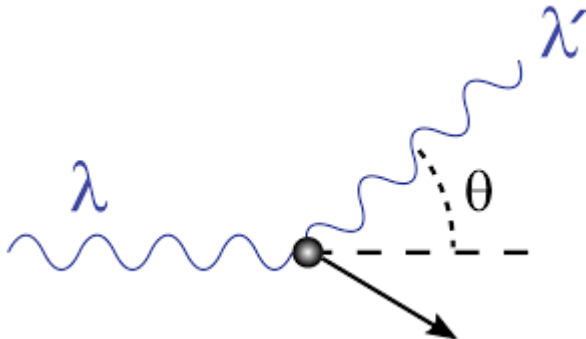


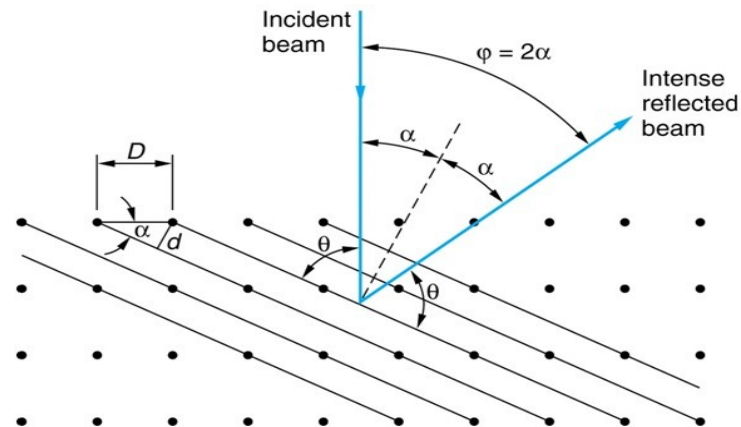
Photo electric effect (1886)



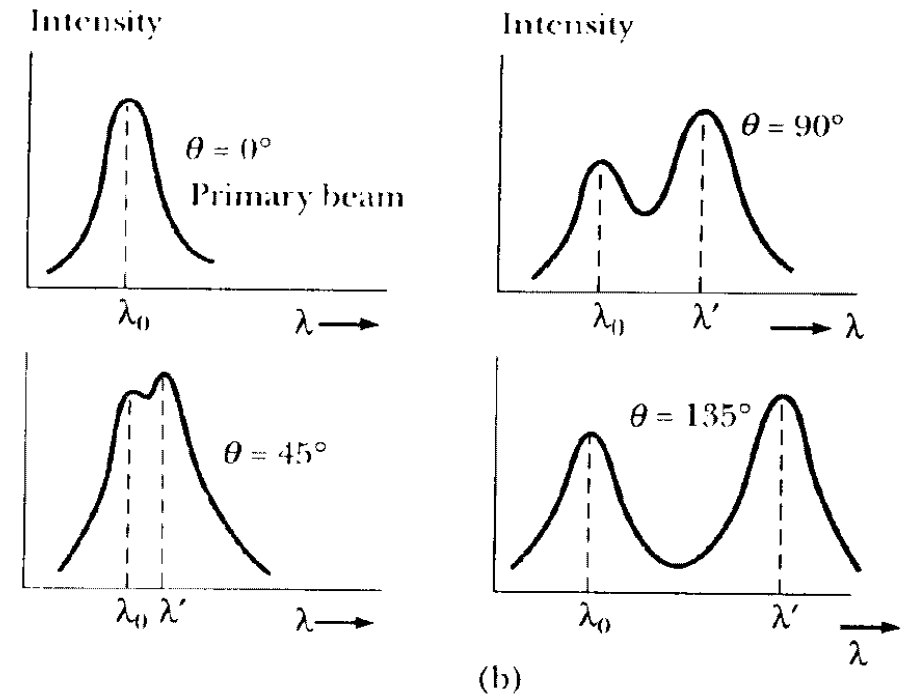
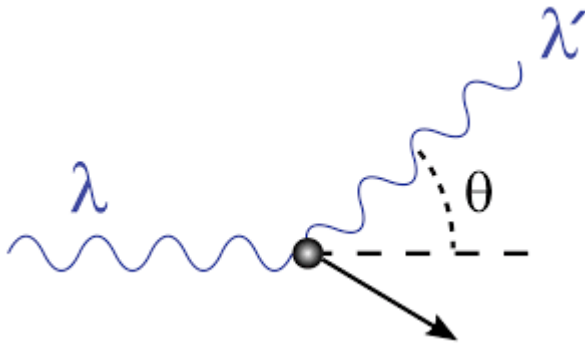
Compton effect (1923)



Davisson-Germer experiment (1927)



Compton effect (1923)



Energy:
$$h\nu + m_0c^2 = E_e + h\nu'$$

$$E_e^2 = p_e^2 c^2 + m_0^2 c^4$$

$$\frac{h\nu}{c} = \frac{h\nu'}{c} \cos \theta + p_e \cos \varphi$$

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0 c^2} (1 - \cos \theta)$$

Momentum

$$\frac{h\nu'}{c} \sin \theta = p_e \sin \varphi$$

$$\Delta \lambda = \lambda' - \lambda = \frac{h}{m_0 c} (1 - \cos \theta)$$

Wave function related to probability amplitude

Ψ

$$P(x, t) = |\psi(x, t)|^2 \quad \psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int d^3 p \quad \varphi(p) \quad e^{\frac{i}{\hbar}(px - E(p)t)}$$

$$P(x, t) dx = |\psi(x, t)|^2 dx$$

Moment we say probability $P(x, t)$

ψ

Should be single valued, at each space point