* The eigen functions of Hermitian operators are Complete, i.e.

$$\psi(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

normalizable

Where $\boxed{a_n = \int u_n^* \psi \, dx}$

Proof

$$\sum_{m=1}^{\infty} \int u_n^* a_m u_m \, dx = \sum_{m=1}^{\infty} a_m \delta_{mn} = a_n$$

Now total probability $\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$

Hence, $$\int_{-\infty}^{\infty} \psi^* \psi \, dx = \sum_m \sum_n \int_{-\infty}^{\infty} a_m^* a_n u_m^* u_n \, dx$$

$$= \sum_m \sum_n a_m^* a_n \delta_{mn} = \sum_{n=1}^{\infty} |a_n|^2$$

$$\therefore \sum_{n=1}^{\infty} |a_n|^2 = 1$$

Probability of measuring the Physical state associated with eigen function u_n.

Particle in an infinite potential Well

Suppose we have a single particle of mass *m* confined to within a region with the following specification ,

$$V(x) = 0 \quad \text{for} \quad 0 < x < L$$
$$= \infty \quad \text{for} \quad x > L \ \& \ x < 0$$

This simple model is useful to describe (in one dimension), the properties of the gas particles in a gas or the properties of a conduction electron within a metal.

Our goal is to learn about the properties of the particle on the basis of what we have learned so far.

The boundary conditions are the following $\psi(0,t) = \psi(L,t) = 0$

This is primarily because the particle can not be found in the regions **x > L or x < 0**

Between the barriers, the energy of the particle is purely kinetic. Let's the total energy is **E** , therefore,

$$\boxed{\frac{P^2}{2m} = E}$$

The time independent Schroedinger equation becomes: $\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi}$ — (1)

Eq. (1) can also be written as $\frac{d^2\psi}{dx^2} + k^2\psi = 0$, where $k = \sqrt{\frac{2mE}{\hbar^2}}$ — (2)

The most general solution for eq^n (2) is given by

$$\psi = A' e^{ikx} + B' e^{-ikx}$$
$$= A\cos kx + B\sin kx$$

Now applying the boundary Cond^n we will get

$\psi(0) = A = 0$ and $\psi(L) = B\sin kL = 0$

Since $B \neq 0$ $\therefore$ $\sin kL = 0$ $\therefore$ $kL = n\pi$

$$\boxed{\therefore k_n = \frac{n\pi}{L}} \quad n = 1, 2, 3, \ldots$$

we exclude $n = 0$, then the wave function will not exists.

$$\therefore \boxed{E_n = \frac{k_n^2\hbar^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}} \Rightarrow \text{The energy is quantized.}$$

Now $\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = B^2 \int_0^L \sin^2 k_n x \, dx = B^2 \times \frac{L}{2} = 1$

$$\boxed{\therefore B = \sqrt{\frac{2}{L}}}$$

$$\therefore \quad \psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \qquad \text{for } 0 < x < L$$

$$= 0 \qquad \text{for } x < 0,\ x > L$$

Also, note that the particle can only have energies E_n, and in particular, the lowest energy, E_1 is greater than zero, as required by uncertainty principle.

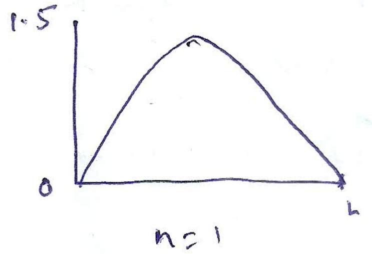
Energy eigenvalues and eigenfunctions

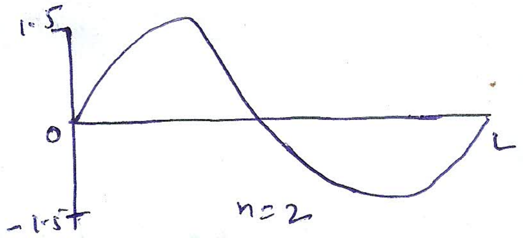
The above wave functions can be written in the form

Energy eigen function

$$\Psi_n(x,t) = \psi_n(x)\, e^{-\frac{iE_n t}{\hbar}}$$

↳ The time dependence of the wave function for any system in a state of given energy is always of this form.

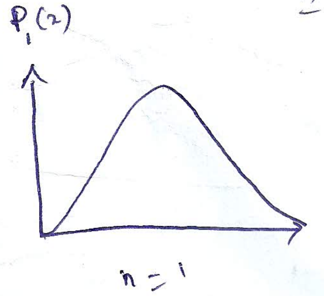


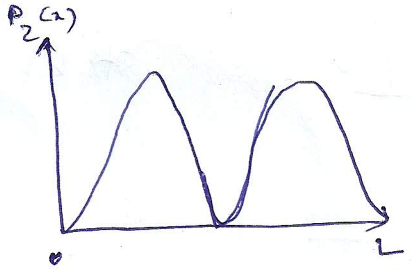


Probability distribution for Position

$$P_n(x) = |\Psi_n(x,t)|^2 = \frac{2}{L} \sin^2 \frac{n\pi x}{L} \qquad 0 < x < L$$

$$= 0 \qquad x < 0,\ x > L$$





For very large 'n' the probability oscillates very rapidly, averaging out to be $\frac{1}{L}$. Particle is equally likely to be found anywhere in the well.

The expectation value of the Position

$$\langle x \rangle = \frac{2}{L} \int_0^L dx\, x \sin^2 \frac{n\pi x}{L} = \frac{L}{2}$$

It is in the middle of the well.

Probability distribution is symmetric about $\frac{L}{2}$

In fact, for, say $n=2$, the particle is most likely to be found in the vicinity of $x = \frac{L}{4}$ and $x = \frac{3L}{4}$.

Now we can calculate

$$\langle x^2 \rangle = \frac{2}{L}\int_0^L x^2 \sin^2\frac{n\pi x}{L}\, dx = L^2 \frac{2n^2\pi^2 - 3}{6n^2\pi^2}$$

∴ The uncertainty in position is

$$(\Delta x)^2 = L^2\frac{2n^2\pi^2 - 3}{6n^2\pi^2} - \frac{L^2}{4} = L^2 \frac{n^2\pi^2 - 6}{12 n^2\pi^2}$$

Orthonormality

$$\int_{-\infty}^{\infty} \psi_m^*(x)\,\psi_n(x)\, dx = \frac{2}{L}\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\, dx = \delta_{mn}$$

$\delta_{mn} = 1$ if $m = n$ ⇒ normalization Cond^n

$= 0$ if $m \neq n$ ⇒ orthogonal cond^n.

Linear Superpositions

Let us consider the superposition of two such wave functions representing the particle having energies E_1 and E_2.

$$\therefore \psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x,t) + \psi_2(x,t)\right]$$

Normalized to unity.

$$= \frac{1}{\sqrt{L}}\left[\sin\frac{\pi x}{L}\, \cancel{e^{-iEt}}\, e^{-i\omega_1 t} + \sin\frac{2\pi x}{L}\, e^{-i\omega_2 t}\right] \quad 0 < x < L$$

$$= 0 \quad x < 0 \text{ and } x > L$$

The probability distribution is

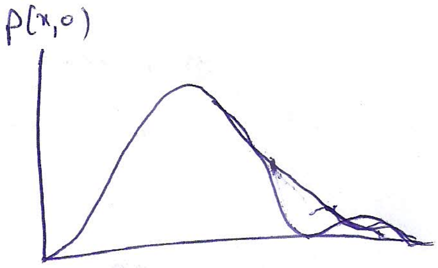
$$P(x,t) = |\psi(x,t)|^2 = \frac{1}{L}\left[\sin^2\frac{\pi x}{L} + \sin^2\frac{2\pi x}{L} + 2\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\cos(\Delta\omega\, t)\right]$$

$$\boxed{\text{Where } \Delta\omega = \frac{E_2 - E_1}{\hbar}}$$

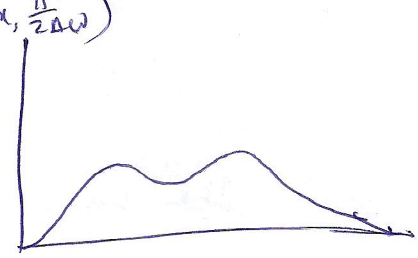
Time dependent Probability distribution.

At $\underline{t=0}$

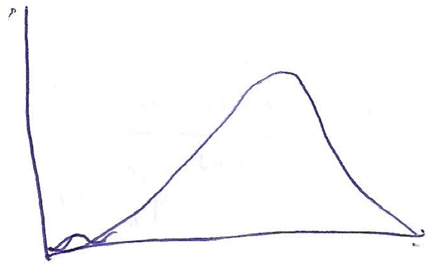
$$P(x,0) = \frac{1}{L}\left(\sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L}\right)^2$$



$$P\left(x, \frac{\pi}{2\Delta\omega}\right) = \frac{1}{L}\left(\sin^2\frac{\pi x}{L} + \sin^2\frac{2\pi x}{L}\right)$$



$$P\left(x, \frac{\pi}{\Delta\omega}\right) = \frac{1}{L}\left(\sin\frac{\pi x}{L} - \sin\frac{2\pi x}{L}\right)^2$$



The maximum probability swings from left to the right hand side of the well but without the max^m moving through the Centre of the well

⇓

Classical ~~analouge~~ behaviour of the particle bouncing back ~~and~~ and forth between the walls.

The total probability of finding the particle on the left hand half of the well,

$$P_L = \int_0^{\frac{L}{2}} P(x,t)\,dx = \frac{1}{2} + \frac{\cos(\Delta\omega t)}{\pi}$$

$$\therefore\ P_R = \frac{1}{2} - \frac{\cos(\Delta\omega t)}{\pi}$$

which illustrates the 'See-sawing' of the probability from one side to the other side. with frequency $(2\pi\Delta\omega)$.

Particle momentum

We can write down $\psi_n(x,t) = \sqrt{\frac{2}{L}}\ \dfrac{e^{i(k_n x - \omega_n t)} - e^{-i(k_n x + \omega_n t)}}{2i}$

$\therefore$ It is associated with particle having momentum $P_n = \hbar k_n$, the other with the particle having momentum $P_n = -\hbar k_n$. Both of them have equal weight.

Here, momentum has a probability distribution which is peaked at the values $\boxed{P_n = \pm \hbar k_n}$

$$\langle P \rangle = \frac{1}{2}\hbar k_n + \frac{1}{2}(-\hbar k_n) = 0$$

$$\langle P^2 \rangle = \frac{1}{2}(\hbar k_n)^2 + \frac{1}{2}(-\hbar k_n)^2 = \hbar^2 k_n^2$$

$$\therefore (\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2 = \hbar^2 k_n^2 = \hbar^2 \left(\frac{n\pi}{L}\right)^2$$

$$\therefore (\Delta x)^2 (\Delta P)^2 = L^2 \frac{n^2\pi^2 - 6}{12 n^2 \pi^2}\ \hbar^2 \left(\frac{n\pi}{L}\right)^2 = \frac{\hbar^2}{4}\left(\frac{n^2\pi^2 - 6}{3}\right)$$

$$\geq \frac{\hbar^2}{4}$$

$$\boxed{\therefore \Delta x\, \Delta P \geq \frac{\hbar}{2}}$$