SOLVABLE LIE ALGEBRAS

A Project Report Submitted
in Partial Fulfilment of the Requirements
for the Degree of

MASTER OF SCIENCE

in

Mathematics and Computing

by

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to the

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April 2014

CERTIFICATE

This is to certify that the work contained in this report entitled **"SOLVABLE LIE ALGEBRAS"** submitted by **BHAGGYADHAR PAHAR** (**Roll No: 122123010**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

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ABSTRACT

In mathematics, a Lie algebra is solvable if its derived series terminates in the zero subalgebra. The most natural example of a solvable lie algebra is the set of all upper triangular $n \times n$ matrices over an algebraically closed field of characteristic zero. Let V be a finite dimensional vector space over an algebraically closed field of characteristic 0. Let $gl(V)$ denote the set of all linear maps from V to itself. The main aim of this project is to understand that given any *solvable* lie subalgebra L of $gl(V)$, there exists a basis of V with respect to which every element of L is represented by an upper triangular matrix.

Contents

Chapter 1

Lie Algebras

In this thesis, F denotes an arbitrary field.

1.1 Introduction

Lie algebras arise "in nature " as vector spaces of linear transformations endowed with a new operation which is in general neither commutative nor associative: $[x, y] = xy - yx$ (where the operations on the right side are the usual ones).It is possible to describe this kind of system abstractly in a few axioms.

1.2 The Notion of a Lie Algebra

Definition 1.2.1. A vector space L over a field F, with an operation $L \times L \rightarrow L$,denoted $(x, y) \mapsto [xy]$ and called the **bracket** or **commutator** of x and y,is called a **Lie algebra** over F if the following axioms are satisfied :

(L1): The bracket operation is bilinear.

(L2): $[xx] = 0 \ \forall x \in L$.

(L3): $[x[yz]] + [y[zx]] + [z[xy]] = 0, (x, y, z \in L)$(**Jacobi Identity**).

Result: Notice that **(L1)** and **(L2)** is applied to $[x+y, x+y]$,imply **anti-commutativity: (L2)'** $[xy] = -[yx]$.

Conversely if $char(F) \neq 2$,it is clear that **(L2)'** will imply **(L2)**.

1.3 Sub-algebra of a Lie algebra

Definition 1.3.1. Let L be a lie algebra. A vector subspace K of L is called a **sub-algebra** if $[xy] \in K$ whenever $x, y \in K$; in particular K is a Lie algebra in its own right relative to the inherited operations.

1.4 Linear Lie algebras

Let V be a finite dimensional vector space over a field F. Let $End(V)$ denote the set of all linear transformations :$V \rightarrow V$. As a vector space over F, $End(V)$ has dimension n^2 $(n = dim(V))$, and $End(V)$ is a ring relative to the usual product operation. Define a new operation $[x, y] = xy - yx$, called the **bracket** of x and y. It is easy to check that with this new operation (called bracket), $End(V)$ becomes a Lie algebra over F. In order to distinguish this new algebra structure from the old associative one, we write $gl(V)$ for $End(V)$ viewed as a Lie algebra and call it the **general linear algebra**.

Definition 1.4.1. Any subalgebra of the Lie algebra $gl(V)$ is called a **linear Lie algebra**.

If we fix a basis for V, then $gl(V)$ can be identified with the set $gl(n, F)$ of all $n \times n$ matrices over F. The standard basis for $gl(n, F)$ is the set of all matrices e_{ij} (having 1 in the (i, j)-th position and 0 elsewhere), where $1 \leq i, j \leq n$. Since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows that the multiplication table for $gl(n, F)$ with respect to its standard basis is given by:

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}.$$

1.5 Examples of Lie algebras

Let us now study some examples of lie algebras, which are central to the theory we are going to develop. They fall into 4 families A_l, B_l, C_l, D_l $(l \geq 1)$ and are called the **classical algebras**. For B_l and D_l, let char $F \neq 2$.

A_l : Let $dimV = l + 1$. Denote by $sl(V)$, or $sl(l + 1, F)$, the set of endomorphisms of V having trace zero.

Since $Tr(xy) = Tr(yx)$ and $Tr(x+y) = Tr(x)+Tr(y)$, it follows that $sl(V)$ is a subalgebra of $gl(V)$, called the **special linear algebra**. Since $sl(V)$ is a proper subalgebra of $gl(V)$, it follows that the dimension of $sl(V)$ is at most $(l + 1)^2 - 1$. On the other hand, we can exhibit this number of linearly independent matrices of trace zero:

Take all $e_{ij}(i \neq j)$, along with all $h_i = e_{ii} - e_{i+1,i+1}(1 \leq i \leq l)$, for a total of $l + (l + 1)^2 - (l + 1)$ matrices. We shall always view this as the standard basis for $sl(l+1,F)$.

C_l : Let $dimV = 2l$, with basis $(v_1, ..., v_{2l})$.
Define a nondegenerate skew-symmetric form f on V by the matrix $s = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$. Let us denote by $sp(V)$,or $sp(2l,F)$ the **symplectic algebra**, which by definition consists of all endomorphisms x of V satisfying $f(x(v),w) = -f(v,x(w))$. It can be easily checked that $sp(V)$ is closed under the bracket operation. In matrix terms, the condition for $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ $(m,n,p,q \in gl(l,F))$ to be symplectic is that $sx = -x^t s$(x^t=transpose of x),i.e., that $n^t = n, p^t = p$, and $m^t = -q$. This last condition forces $Tr(x) = 0$.
It is easy now to compute a basis for $sp(2l,F)$.
Take the diagonal matrices $e_{ii} - e_{l+i,l+i}(1 \leq i \leq l), l$ in all. Add to these all $e_{ij} - e_{l+j,l+i}(1 \leq i \neq j \leq l), l^2 - l$ in number. For n we use the matrices $e_{i,l+i}(1 \leq i \leq l)$ and $e_{i,l+j} + e_{j,l+i}(1 \leq i < j \leq l)$, a total of $l + \frac{1}{2}l(l-1)$, and similarly for the positions in p. Adding up, we find $dim\ sp(2l,F) = 2l^2 + l$.

B_l : Let $dimV = 2l+1$ be odd, and take f to be the nondegenerate symmetric bilinear form on V whose matrix is $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$. The **orthogonal algebra** $o(V)$, or $o(2l+1,F)$,consists of all endomorphisms x of V satisfying $f(x(v),w) = -f(v,x(w))$ (the same requirement as for

C_l). If we partition x in the same form as s, say $x=\begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$, then the condition $sx = -x^t s$ translates into the following set of conditions: $a = 0, c_1 = -b_2^t, c_2 = -b_1^t, q = -m^t, n^t = -n, p^t = -p$. As in the case of C_l, this shows that $Tr(x) = 0$.

For a basis, take first the l diagonal matrices $e_{ii} - e_{l+i,l+i}$ $(2 \leq i \leq l+1)$. Add the $2l$ matrices involving only row one and column one: $e_{1,l+i+1} - e_{i+1,1}$ and $e_{1,i+1} - e_{l+i+1,1}(1 \leq i \leq l)$. Corresponding to $q = -m^t$, take $e_{i+1,j+1} - e_{l+j+1,l+i+1}(1 \leq i \neq j \leq l)$. For n take $e_{i+1,l+j+1} - e_{j+1,l+i+1}(1 \leq i < j \leq l)$, and for p, $e_{i+l+1,j+1} - e_{j+l+1,i+1}(1 \leq j < i \leq l)$. The total number of basis elements is $2l^2 + l$.

$D_l(l \geq 2)$: Here we obtain another **orthogonal algebra**. The construction is identical to that for B_l, except that $dim\ V = 2l$ is even and s has the simpler form $\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$. We can show similarly (as in the above cases) that $dim\ o(2l, F) = 2l^2 - l$.

Let $t(n, F)$ be the set of **upper triangular matrices** (a_{ij}), $a_{ij} = 0$ if $i > j$. Let $n(n, F)$ be the **strictly upper triangular matrices** ($a_{ij} = 0$ if $i \geq j$). Finally, let $\delta(n, F)$ be the set of all **diagonal matrices**. It is trivial to check that each of $t(n, F), n(n, F)$ and $\delta(n, F)$ is closed under the bracket. Hence each of these is a sub-algebra of $gl(n, F)$.

Notice also that $t(n, F) = \delta(n, F) + n(n, F)$ (vector space direct sum), with $[\delta(n, F), n(n, F)] = n(n, F)$, hence $[t(n, F), t(n, F)] = n(n, F)$. (Where if

H, K are subalgebras of L, then $[HK]$ denotes the subspace of L spanned by commutators $[xy]$,$x \in H, y \in K$.)

1.6 Lie algebras of derivations

Definition 1.6.1. By an $F-$**algebra** (not necessarily associative), we simply mean a vector space U over F endowed with a bilinear opearation $U \times U \to U$, usually denoted by juxtaposition.

Definition 1.6.2. By a **derivation** of U, we mean a linear map $\delta : U \to U$ satisfying the familiar product rule $\delta(ab) = a\delta(b) + \delta(a)b$.

It can be easily checked that collection *Der* U of all derivations of U is a vector subspace of *End* U.

Result: The commutator $[\delta, \delta']$ of two derivations is again a derivation.

Proof: Clearly the commutator $[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$ is a linear map from U into U. It therefore suffices to check that the commutator $[\delta, \delta']$ satisfies the product rule:

$$[\delta, \delta'](ab) = a[\delta, \delta'](b) + b[\delta, \delta'](a)$$

But this is obvious since $[\delta, \delta'](ab) = (\delta \circ \delta')(ab) - (\delta' \circ \delta)(ab) = \delta(\delta'(ab)) - \delta'(\delta(ab)) = \delta(a\delta'(b) + b\delta'(a)) - \delta'(a\delta(b) + b\delta(a)) = a\delta(\delta'(b)) + \delta(a)\delta'(b) + \delta'(a)\delta(b) + b\delta(\delta'(a)) - a\delta'(\delta(b)) - \delta(b)\delta'(a) - b\delta'(\delta(a)) - \delta(a)\delta'(b) = a[\delta \circ \delta'(b) - \delta' \circ \delta(b)] + b[\delta \circ \delta'(a) - \delta' \circ \delta(a)] = a[\delta, \delta'](b) + b[\delta, \delta'](a)$.

Hence *Der* U is a subalgebra of $gl(U)$.

Adjoint representation : Since a Lie algebra L is an F-algebra in the above sense, $Der\ L$ is defined. Certain derivations arise quite naturally, as follows. If $x \in L$,$y \mapsto [xy]$ is an endomorphism of L, which we denote $ad\ x$. In fact ,$ad\ x \in Der\ L$ because we can rewrite the Jacobi identity in the form:$[x[yz]] = [[xy]z] + [y[xz]]$. Derivations of this form are called **inner**, all others **outer**.
The map $L \to Der\ L$ sending x to $ad\ x$ is called the **adjoint representation** of L.

1.7 Abstract Lie algebras

If L is an arbitrary finite dimensional vector space over F, we can view L as a Lie algebra by setting $[xy] = 0$ for all $x, y \in L$. Such an algebra, having trivial Lie multiplication ,is called **abelian** (because in the linear case $[x, y] = 0$ just means that x and y commute). If L is any Lie algebra,with basis $x_1, ..., x_n$, it is clear that the entire multiplication table of L can be recovered from the **structure constants** a_{ij}^k which occur in the expressions $[x_i x_j] = \Sigma_{k=1}^n a_{ij}^k x_k$.
Those for which $i \geq j$ can even be deduced from the others, thanks to (L2), (L2'). Turning this remark around, it is possible to define an abstract Lie algebra from scratch simply by specifying a set of structure constants. A moment's thought shows that it is enough to require the "obvious" identities, those implied by (L2) and (L3):

$$a_{ii}^k = 0 = a_{ij}^k + a_{ji}^k$$

$$\Sigma_k (a_{ij}^k a_{kl}^m + a_{jl}^k a_{ki}^m + a_{li}^k a_{kj}^m) = 0$$

.

As an application of the abstract point of view, we can determine (up to isomorphism) all Lie algebras of dimension ≤ 2. In dimension 1 there is a single basis vector x, with multiplication table $[xx] = 0$**(L2)**. In dimension 2, start with a basis x, y of L. Clearly, all products in L yield scalar multiples of $[xy]$. If these are all 0, then L is abelian. Otherwise, we can replace x in the basis by a vector spanning the one dimensional space of multiples of the original $[xy]$, and take y to be any other vector independent of the new x. Then $[xy] = ax(a \neq 0)$. Replacing y by $a^{-1}y$, we finally get $[xy] = x$. Abstractly, therefore, at most one nonabelian L exists and the equation $[xy] = x$ characterizes all non-abelian lie algebras of dimension 2.

Chapter 2

Ideals and Homomorphisms

2.1 Ideals

Definition 2.1.1. A subspace I of a Lie algebra L is called an **ideal** of L if $x \in L$, $y \in I$ together imply $[xy] \in I$.

2.1.1 Examples of Ideals

1. 0 and L itself are ideals of L.

2. **Center** $Z(L)$=$\{z \in L | [xz] = 0, \forall x \in L\}$. Notice that L is abelian if and only if $Z(L) = L$.

3. **Derived algebra** of L, denoted $[LL]$, which is analogous to the commutator subgroup of a group. It consists of all linear combinations of commutators $[xy]$, and is clearly an ideal.

Result: L is abelian if and only if $[LL] = 0$.

Result: If I, J are two ideals of a Lie algebra L, then $I + J$:=$\{x + y | x \in I, y \in J\}$is also an ideal.

Similarly, $[IJ]$=$\{\Sigma[x_iy_i]|x_i \in I, y_i \in J\}$ is an ideal.

Definition 2.1.2. If L has no ideals except itself and 0, and if moreover $[LL] \neq 0$, we call L **simple**.

Result: L simple implies $Z(L) = 0$ and $L = [LL]$.

Example: Let $L = sl(2, F)$,$char\ F \neq 2$.

Take as standard basis for L the three matrices :

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The multiplication table is then completely determined by the equations: $[xy] = h, [hx] = 2x, [hy] = -2y$.(Notice that x, y, h are eigenvectors for $ad\ h$, corresponding to the eigenvalues $2, -2, 0$. Since $char\ F \neq 2$, these eigenvalues are distinct). If $I \neq 0$ is an ideal of L, let $ax + by + ch$ be an arbitrary nonzero element of I. Applying $ad\ x$ twice, we get $-2bx \in I$, and applying $ad\ y$ twice, we get $-2ay \in I$. Therefore, if a or b is nonzero, I contains either y or x ($char\ F \neq 2$), and then, clearly,$I = L$ follows. On the other hand, if $a = b = 0$, then $0 \neq ch \in I$, so $h \in I$, and again $I = L$ follows. We conclude that L is simple.

Quotient algebra: The construction of a **quotient algebra** L/I (I an ideal of L) is formally the same as the construction of a quotient ring: as vector space L/I is just the quotient space, while its Lie multiplication is defined by $[x+I, y+I]$=$[xy]+I$. This is unambiguous, since if $x+I = x'+I$ and $y + I = y' + I$, then we have $x' = x + u(u \in I)$ and $y' = y + v(v \in I)$, whence $[x'y'] = [xy] + ([uy] + [xv] + [uv])$, and therefore $[x'y'] + I = [xy] + I$, since the terms in parentheses all lie in I.

2.2 Homomorphisms and representations

Definition 2.2.1. A linear transformation $\phi : L \to L'$ (L, L' Lie algebras over F) is called a **homomorphism** if $\phi([xy]) = [\phi(x)\phi(y)]$, for all $x, y \in L$.

ϕ is called a **monomorphism** if $Ker\ \phi = 0$, an **epimorphism** if $Im\, \phi = L'$, an **isomorphism** if it is both monomorphism and epimorphism.

Result: $Ker\ \phi$ is an ideal of L.

Result: $Im\ \phi$ is a subalgebra of L'.

As in other algebraic theories, there is a natural $1-1$ correspondence between homomorphisms and ideals: to ϕ is associated $Ker\phi$, and to an ideal I is associated the **canonical map** $x \mapsto x + I$ of L onto L/I. The following standard homomorphism theorems can be easily verified:

Theorem 2.2.2. *If $\phi : L \to L'$ is a homomorphism of Lie algebras, then $L/Ker\ \phi \cong Im\ \phi$. If I is any ideal of L included in $Ker\ \phi$, there exists a unique homomorphism $\psi : L/I \to L'$ such that $\phi = \psi \circ \pi$ where $\pi : L \to L/I$ is the canonical map.*

Theorem 2.2.3. *If I and J are ideals of L such that $I \subset J$, then J/I is an ideal of L/I and $(L/I)/(J/I)$ is naturally isomorphic to (L/J).*

Theorem 2.2.4. *If I, J are ideals of L, there is a natural isomorphism between $(I + J)/J$ and $I/(I \cap J)$.*

A **representation** of a Lie algebra L is a Lie algebra homomorphism $\phi : L \to gl(V)$ (V= vector space over F). The adjoint representation is the lie algebra homomorphism ad:$L \to gl(L)$ which sends x to $ad\ x$, where $ad\ x(y) = [xy]$. It is clear that ad is a linear transformation. To see it preserves the bracket, we calculate:

$$[ad\ x, ad\ y](z) = ad\ x\ ad\ y(z) - ad\ y\ ad\ x(z)$$

$$= ad\ x([yz]) - ad\ y([xz]) = [x[yz]] - [y[xz]]$$

$$= [x[yz]] + [[xz]y] = [[xy]z] = ad[xy](z).$$

What is the *kernel* of ad ? It consists of all $x \in L$ for which $ad\ x = 0$, i.e., for which $[xy] = 0$ (all $y \in L$). So $Ker\ ad = Z(L)$. If L is simple, then $Z(L) = 0$, so that ad:$L \to gl(L)$ is a monomorphism. This means that any simple Lie algebra is isomorphic to a linear Lie algebra.

2.3 The invariance lemma

Definition 2.3.1. A **weight** for a lie subalgebra A of $gl(V)$ is a linear map $\lambda : A \to F$ such that

$$V_\lambda = \{v \in V : a(v) = \lambda(a)v\ \forall\ a \in A\}$$

is a non-zero subspace of V.

Lemma (The invariance lemma): Assume that F has characteristic zero. Let V be a finite dimensional vector space over F. Let L be a lie subalgebra of $gl(V)$ and let A be an ideal of L. Let $\lambda : A \to F$ be a weight of A. The associated weight space

$$V_\lambda = \{v \in V : av = \lambda(a)v \forall a \in A\}$$

is an L-invariant subspace of V.

Proof: We must show that if $y \in L$ and $w \in V_\lambda$, then $y(w)$ is an eigenvector for every element of A, with the eigenvalue of $a \in A$ given by $\lambda(a)$. For $a \in A$, we have

$$a(yw) = y(aw) + [a, y](w) = \lambda(a)yw + \lambda([a, y])w.$$

Note that $[a, y] \in A$ as A is an ideal.Therefore all we need to show that the eigenvalue of the commutator $[a, y]$ on V_λ is zero.

Consider U=$Span\{w, y(w), y^2(w), ...\}$. This is a finite -dimensional subspace of V. Let m be the least number such that the vectors $w, y(w), \ldots, y^m(w)$ are linearly dependent. It is a straightforward exercise in linear algebra to show that U is m-dimensional and has as a basis

$$w, y(w), ..., y^{m-1}(w).$$

We claim that if $z \in A$, then z maps U into itself. In fact, we will show more than this claim, namely we will show that with respect to the basis above, z has the form of an upper triangular matrix with diagonal entries equal to

$\lambda(z)$:

$$\begin{pmatrix} \lambda(z) & \star & \cdots & \star \\ 0 & \lambda(z) & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda(z) \end{pmatrix}$$

We work by induction on the number of the column. First of all, $zw = \lambda(z)w$. This gives the first column of the matrix. Next ,since $[z, y] \in A$, we have

$$z(yw) = y(zw) + [z, y]w = \lambda(z)y(w) + \lambda([z, y])w$$

giving the second column.

For column r, we have

$$z(y^r(w)) = zy(y^{r-1}w) = (yz + [z, y])y^{r-1}w.$$

By the induction hypothesis , we can say that

$$z(y^{r-1}w) = \lambda(z)y^{r-1}w + u$$

for some u in the span of $\{y^j w : j < r - 1\}$. Substituting this gives

$$yz(y^{r-1}w) = \lambda(z)y^r w + yu$$

and yu belongs to the span of the $\{y^j w : j < r\}$. Furthermore, since $[z, y] \in A$, we get by induction that $[z, y]y^{r-1}w = v$ for some v in the span of

$\{y^j w : j \leq r-1\}$. Combining the last two results shows that column r is as stated.

Now take $z = [a, y]$. We have just shown that the trace of the matrix of z acting on U is $m\lambda(z)$. On the other hand, by the previous paragraph, U is invariant under the action of $a \in A$, and U is y-invariant by construction. So the trace of z on U is the trace of $ay - ya$, also viewed as a linear transformation of U, and this is obviously 0. Therefore $m\lambda([a, y]) = 0$.

As F has characteristic zero , it follows that $\lambda([a, y]) = 0$.

Chapter 3

Solvable Lie algebras

3.1 Solvability

Definition 3.1.1. Let L be a lie algebra. The **derived series** of L is defined by a sequence of ideals, $L^{(0)} = L$, $L^{(1)} = [LL]$, $L^{(2)} = [L^{(1)}, L^{(1)}]$, ..., $L^{(i)} = [L^{(i-1)}L^{(i-1)}]$. L is called **solvable** if $L^{(n)} = 0$ for some n.

For example, abelian implies solvable, whereas simple lie algebras are definitely nonsolvable.

Example: The lie algebra $L = t(n, F)$ of upper triangular $n \times n$ matrices over F is solvable.

Proof: Let $n(n, F)$ denote the set of all strictly upper traingular $n \times n$ matrices over F. Clearly, $[t(n, F), t(n, F)] \subseteq n(n, F)$. On the other hand, the standrad basis for $n(n, F)$ consists of the matrices e_{il} where $i < l$ and e_{il} is the $n \times n$ matrix over F having 1 in the (i, l)-th position and 0 elsewhere. Clearly, the commutator $[e_{ii}, e_{il}] = e_{il}$ for $i < l$ where e_{ii} denotes the $n \times n$ matrix over F having 1 in the (i, i)-th position and 0 elsewhere. This implies that $n(n, F) \subseteq [t(n, F), t(n, F)]$. Hence we have $L^{(1)} = n(n, F)$.

For $i < j$, let us define the *level* of e_{ij} to be the integer $j - i$. Then $L^{(2)}$ is spanned by commutators of the form $[e_{ij}, e_{kl}]$ where $i < j$ and $k < l$. But observe that for $i < j$ and $k < l$,

$$[e_{ij}, e_{kl}] = \begin{cases} -e_{kj} & \text{if } i = l \\ e_{il} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

Notice that if $i = l$, then $j \neq k$ and the level of $e_{kj} = j - k = (j - i) + (l - k) \geq 2$ (since $i < j$ and $k < l$, we have $j - i \geq 1$ and $l - k \geq 1$). Similarly, notice that if $j = k$, then $i \neq l$ and the level of $e_{il} = l - i = (l - k) + (j - i) \geq 2$. Hence $L^{(2)}$ is spanned by e_{rs} where the level of e_{rs} is at least 2. One can easily prove using induction on i that $L^{(i)}$ is spanned by e_{rs} of level at least 2^{i-1}. Since level of any e_{ij} is $\leq n - 1$, it is now clear that $L^{(i)} = 0$ whenever $2^{i-1} > n - 1$. Hence L is solvable.

Proposition: Let L be a Lie algebra.

(a) If L is solvable , then so are all subalgebras and homomorphic images of L.

(b) If I is a solvable ideal of L such that L/I is solvable, then L itself is solvable.

(c) If I, J are solvable ideals of L, then so is $I + J$.

Proof:

(a) From the definition, if K is a subalgebra of L, then $K^{(i)} \subset L^{(i)}$. Similarly, if $\phi : L \to M$ is an epimorphism, an easy induction on i shows that $\phi(L^{(i)}) = M^{(i)}$.

(b) Say $(L/I)^{(n)} = 0$. Applying part (a) to the canonical homomorphism $\pi : L \to L/I$, we get $\pi(L^{(n)}) = 0$, or $L^{(n)} \subset I = Ker\ \pi$. Now if $I^{(m)} = 0$, the obvious fact that $(L^{(i)})^{(j)} = L^{(i+j)}$ implies that $L^{(n+m)} = 0$ (apply proof of part (a) to the situation $L^{(n)} \subset I$).

(c) One of the standard homomorphism theorems (Theorem 2.2.4) yields an isomorphism between $(I + J)/J$ and $I/(I \cap J)$. As a homomorphic image of I, the right side is solvable, so $(I + J)/J$ is solvable. Then so is $I + J$, by part (b) applied to the pair $I + J, J$.

Definition 3.1.2. Let L be an arbitrary Lie algebra. Let S be a maximal solvable ideal of L (that is, one included in no larger solvable ideal). If I is any other solvable ideal of L, then by part (c) of the previous proposition, $I + S$ is solvable and $S \subseteq I + S$. But then, by maximality of S, we have $I + S = S$. Hence $I \subseteq S$. This proves the existence of a unique maximal solvable ideal, called the **radical** of L and denoted $Rad\ L$.

If $Rad\ L = 0$ then L is called **semisimple**. For example, a simple lie algebra is semisimple.

3.2 Lie's theorem

Proposition 1: Let V be a non-zero complex vector space. Suppose that L is a solvable Lie subalgebra of $gl(V)$. Then there is some non-zero $v \in V$ which is a simultaneous eigenvector for all $x \in L$.

Proof: We use induction on dim(L). If dim(L)=1, say $L = span\{x_0\}$. Then

$x_0 : V \to V$ is a linear map, hence it has an eigen vector $v \in V$. It is now easy to see that v is a simultaneous eigen vector for all $x \in L$. So we may assume that dim(L)> 1.

Since L is solvable , we know that $L' = [L, L]$ is properly contained in L. Choose a subspace A of L which contains L' and is such that $L = A \oplus Span\{z\}$ for some $0 \neq z \in L$.

Since $L' \subseteq A$, it follows that A is an ideal of L. Since any ideal of L is a subalgebra of L, therefore A is a subalgebra of L. Since L is solvable and any subalgebra of a solvable algebra is solvable, it follows that A is solvable. We may now apply the inductive hypothesis to obtain a vector $w \in V$ which is a simultaneous eigenvector for all $a \in A$. Let $\lambda : A \to \mathbb{C}$ be the corresponding weight, so $a(w) = \lambda(a)w$ for all $a \in A$. Let V_λ be the weight space corresponding to λ:

$$V_\lambda = \{v \in V : a(v) = \lambda(a)v, \forall a \in A\}.$$

This eigenspace is non-zero, as its contains w. By the Invariance Lemma, the space V_λ is L-invariant. So $z \in L$ is a linear transformation which takes V_λ into itself. Hence there is some non-zero $v \in V_\lambda$ which is an eigenvector for z.

We claim that v is a simultaneous eigenvector for all $x \in L$. Any $x \in L$ may be written in the form $x = \alpha + \beta z$ for some $a \in A$ and $\beta \in \mathbb{C}$. We have

$$x(v) = a(v) + \beta z(v) = \lambda(a)v + \beta\mu v$$

where μ is the eigenvalue of z corresponding to v. This completes the proof.

Theorem 3.2.1. ***Lie's theorem:*** *Let V be an n-dimensional complex vector space and let L be a solvable Lie subalgebra of $gl(V)$. Then there is a basis of V in which every element of L is represented by an upper triangular matrix.*

Proof: It follows from the previous proposition that there exists a non-zero vector $v \in V$ such that $x(v) = \lambda(x)v$ for all $x \in L$. Let $U = Span\{v\}$ and let $\bar{V}$ be the quotient vector space V/U. Any $x \in L$ induces a linear transformation $\bar{x}$ of $\bar{V}$, namely:

$$\bar{x} : \bar{V} \to \bar{V} \textit{ is given by}$$

$$\bar{x}(w + U) := x(w) + U.$$

It can be easily checked that the map $L \to gl(\bar{V})$ given by $x \mapsto \bar{x}$ is a Lie algebra homomorphism. The image of L under this homomorphism is a solvable subalgebra of $gl(\bar{V})$. Moreover, $dim(\bar{V}) = n - 1$, where $n = dim(V)$. So by induction, there exists a basis of $\bar{V}$ such that with respect to this basis, all $\bar{x}$ are upper triangular. If this basis is $\{v_i + U : 1 \leq i \leq n - 1\}$, then $\{v, v_1, \ldots, v_{n-1}\}$ is a basis for V. Since $x(v) = \lambda(x)v$ for each $x \in L$ and since $\{v_i + U : 1 \leq i \leq n - 1\}$ is a basis of $\bar{V}$ with respect to which all $\bar{x}$ are upper triangular, it follows easily that the matrices of all elements of L with respect to the basis $\{v, v_1, \ldots, v_{n-1}\}$ of V are upper triangular.

3.3 Lie's theorem fails in char p

In this section, we will show that Lie's theorem fails in characteristic p fields, where $p > 0$ is a prime. For this purpose, let us consider the following example.

Example: Here we will show that Proposition 1 (and hence Lie's theorem) will fail if the characteristic of the field F is p where p is a prime.

Consider the following $p \times p$ matrices

$$x = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p-2 & 0 \\ 0 & 0 & \cdots & 0 & p-1 \end{pmatrix}$$

It is easy to check that $[x, y] = x$. Hence x and y span a 2-dimensional non-abelian lie subalgebra L of $gl(p, F)$. L is also solvable because $L^{(1)} = [L, L]$ is spanned by the commutator $[x, y]$ and $[x, y] = x$. So $L^{(1)}$ is one dimensional spanned by x alone. Hence $L^{(2)} = [L^{(1)}, L^{(1)}] = 0$, which implies that L is solvable. We will now show that x and y have no common eigenvector.

Here $L \subseteq gl(V)$ where $dim\ V = p$. If Lie's theorem holds true then there exist a non-zero vector $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{pmatrix} \in V$ which is a simultaneous eigen vector for all elements of L. This implies v should be an eigen vector for both of x

and y. That is,

$$xv = \lambda(x)v, yv = \lambda(y)v$$

for some scalars $\lambda(x)$ and $\lambda(y)$.

That means $xv = \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_p \\ v_1 \end{pmatrix} = \lambda(x) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{pmatrix}$ and $yv = \begin{pmatrix} 0 \\ 1.v_2 \\ 2.v_3 \\ \vdots \\ (p-2)v_{p-1} \\ (p-1)v_p \end{pmatrix} = \lambda(y) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{pmatrix}$

Solving these two equations we get

$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$-which is a contradiction since v is non-zero.

Bibliography

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