Lecture 15 & partial of Lecture 16, MA 102

8 & 9 April, 2015

Recall: Singular points

Consider a general linear 2 nd order DE:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \tag{1}$$

Putting this equation in standard form, we get

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

If x_0 is a point (in the complex pane) such that at least one of $P(x)$ or $Q(x)$ is NOT analytic at x_0, then we call x_0 a **singular point** of equation (1).

For the time being assume $x_0 = 0$. Now recall homogeneous **Cauchy-Euler** equations: $ax^2y'' + bxy' + cy = 0$. Clearly $x_0 = 0$ is a singular point of this DE.

Does power series solution work for Cauchy-Euler eqn?

Let us see what we get by plugging in $y = \Sigma_{n=0}^{\infty} c_n x^n$ in the homogeneous Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$ of 2 nd order.
It turns out that we get

$$cc_0 + (b+c)c_1 x + \Sigma_{k=2}^{\infty}[ak(k-1)c_k + kbc_k + cc_k]x^k = 0$$

which implies that if $c \neq 0$ and $b + c \neq 0$ and $ak(k-1) + bk + c \neq 0 \ \forall k \geq 2$, then $c_k = 0 \ \forall k \geq 0$. Thus it is likely that we only get the trivial solution out of this!

What can be done to tackle this situation?

Observe that if we put the above mentioned Cauchy-Euler equation into **standard form**, we get:

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x) = \frac{b}{ax}$ and $Q(x) = \frac{c}{ax^2}$. It is also easy to see that $xP(x)$ and $x^2Q(x)$ are analytic at 0.
This motivates us to consider a **more general** kind of 2 nd order homogeneous linear DE (in standrad form):

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

where $P(x)$ and $Q(x)$ are such that both $xP(x)$ and $x^2Q(x)$ are analytic at 0.
HOW TO SOLVE SUCH EQUATIONS?——We will see this soon.

Two types of Singular points

This motivates the following definition:

DEFINITION 6.2 **Regular and Irregular Singular Points**

A singular point $x = x_0$ of equation (1) is said to be a **regular singular point** if both $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at x_0. A singular point that is not regular is said to be an **irregular singular point** of the equation.

Polynomial Coefficients In the case in which the coefficients in (1) are polynomials with no common factors, Definition 6.2 is equivalent to the following.

Let $a_2(x_0) = 0$. Form $P(x)$ and $Q(x)$ by reducing $a_1(x)/a_2(x)$ and $a_0(x)/a_2(x)$ to lowest terms, respectively. If the factor $(x - x_0)$ appears at most to the first power in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$, then $x = x_0$ is a regular singular point.

Is the above definition good enough?

For example, consider $x^4y'' + x^2 sin\ xy' + (1 - cos\ x)y = 0$. Check that 0 is a **regular singular** point of this DE.

But diff equations of the form $x^{101}y'' + x\alpha(x)y' + \beta(x)y = 0$, where $\alpha(x)$ and $\beta(x)$ are analytic at 0, have been ignored. Why? We could have defined a regular singular point to be a point x_0 such that there exists a positive integer k such that $(x - x_0)^k P(x)$ and $(x - x_0)^{k+1} Q(x)$ are analytic at x_0! But this kind of definition may give rise to lengthy computation.

But still, IT IS NOT A GOOD DEFINITION. PEOPLE MIGHT HAVE BEEN MOTIVATED FROM Cauchy-Euler equation, THATS WHY THEY MADE SUCH A DEFINITION.

Classification of Singular points

EXAMPLE 1 **Classification of Singular Points**

It should be clear that $x = -2$ and $x = 2$ are singular points of the equation

$$(x^2 - 4)^2 y'' + (x - 2)y' + y = 0.$$

Dividing the equation by $(x^2 - 4)^2 = (x - 2)^2(x + 2)^2$, we find that

$$P(x) = \frac{1}{(x-2)(x+2)^2} \quad \text{and} \quad Q(x) = \frac{1}{(x-2)^2(x+2)^2}.$$

We now test $P(x)$ and $Q(x)$ at each singular point.

In order for $x = -2$ to be a regular singular point, the factor $x + 2$ can appear at most to the first power in the denominator of $P(x)$ and can appear at most to the second power in the denominator of $Q(x)$. Inspection of $P(x)$ and $Q(x)$ shows that the first condition does not hold, and so we conclude that $x = -2$ is an irregular singular point.

In order for $x = 2$ to be a regular singular point, the factor $x - 2$ can appear at most to the first power in the denominator of $P(x)$ and can appear at most to the second power in the denominator of $Q(x)$. Further inspection of $P(x)$ and $Q(x)$ shows that both these conditions are satisfied, so $x = 2$ is a regular singular point. ■

Classification of Singular points

EXAMPLE 2 **Classification of Singular Points**

Both $x = 0$ and $x = -1$ are singular points of the differential equation

$$x^2(x+1)^2y'' + (x^2-1)y' + 2y = 0.$$

Inspection of

$$P(x) = \frac{x-1}{x^2(x+1)} \quad \text{and} \quad Q(x) = \frac{2}{x^2(x+1)^2}$$

shows that $x = 0$ is an irregular singular point since $(x-0)$ appears to the second power in the denominator of $P(x)$. Note, however, that $x = -1$ is a regular singular point. ■

Some more interesting examples:

$x = 0$ is a regular singular point of $xy'' - 2xy' + 5y = 0$ since

$$P(x) = -2 \quad \text{and} \quad Q(x) = \frac{5}{x}.$$

How to find a solution at regular singular points?

Also, recall that singular points can be complex numbers. It should be apparent that both $x = 3i$ and $x = -3i$ are regular singular points of the equation $(x^2 + 9)y'' - 3xy' + (1 - x)y = 0$ since

$$P(x) = \frac{-3x}{(x - 3i)(x + 3i)} \quad \text{and} \quad Q(x) = \frac{1 - x}{(x - 3i)(x + 3i)}.$$

Recall the differential equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{3}$$

where $P(x)$ and $Q(x)$ are such that both $xP(x)$ and $x^2Q(x)$ are analytic at 0, that is, 0 is a regular singular point. Since equation (3) was a generalized version of Cauchy-Euler equation (homogeneous and of 2 nd order), let us first see whether we can solve equation (3) using the same technique as we used to solve Cauchy-Euler equations, that is, by putting $y = x^r$ and then determining r.

How to find a solution at regular singular points? contd ...

Multiplying equation (3) by x^2, we get

$$x^2y'' + x(xP(x))y' + (x^2Q(x))y = 0 \tag{4}$$

Now since $xP(x)$ and $x^2Q(x)$ are both analytic at 0, they have power series expansion centered at 0, say, $xP(x) = \Sigma_{n=0}^{\infty} p_n x^n$ and $x^2Q(x) = \Sigma_{n=0}^{\infty} q_n x^n$. Putting $y = x^r$ in equation (4), we get

$$r(r-1)x^r + rx^r(\Sigma_{n=0}^{\infty} p_n x^n) + x^r(\Sigma_{n=0}^{\infty} q_n x^n) = 0$$

which implies

$$[r(r-1) + rp_0 + q_0]x^r + \Sigma_{n=1}^{\infty}(rp_n + q_n)x^{n+r} = 0$$

$$\Rightarrow r(r-1) + rp_0 + q_0 = 0 \; and \; rp_n + q_n = 0 \; \forall n \geq 1.$$

But this is absurd. Hence this method is not helpful.

How to find a solution at regular singular points? contd ...

What about simply using $y = \Sigma_{n=0}^{\infty} c_n x^n$ as a trial solution of (3)? But no, that also does not help, because we can go back to slide 3 and see what happened when we plugged in $y = \Sigma_{n=0}^{\infty} c_n x^n$ in the general 2 nd order homogeneous Cauchy-Euler equation.
Observe that in equation (4) above, the coefficients of y'', y' and y are “Euler coefficients“ times power series. Hence it seems natural to seek solutions of (4) in the form of “Euler solutions“ times power series. Thus we assume that

$$y = x^r \Sigma_{n=0}^{\infty} a_n x^n \qquad (*)$$

is a trial solution (where $a_0 \neq 0$) and see if we can determine some value(s) of r so that $(*)$ is a solution of (4). We have assumed that $a_0 \neq 0$ to guarantee that r is the coefficient of the first term in the series and a_0 is its coefficient.

Solution at regular singular point contd ...

Plugging in $(*)$ in equation (4) and using the power series expansions $xP(x) = \Sigma_{n=0}^{\infty} p_n x^n$ and $x^2 Q(x) = \Sigma_{n=0}^{\infty} q_n x^n$ for $xP(x)$ and $x^2 Q(x)$, we get (after simplification):

$$a_0 F(r) x^r + \Sigma_{n=1}^{\infty} \{F(r+n) a_n + \Sigma_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}]\} x^{r+n} = 0$$

where

$$F(r) = r(r-1) + p_0 r + q_0.$$

So we must have $a_0 F(r) = 0 \Rightarrow F(r) = 0$ (since $a_0 \neq 0$) and

$$F(r+n) a_n + \Sigma_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}] = 0 \; \forall n \geq 1. \qquad (**)$$

The Indicial equation

The equation

$$F(r) = 0 \qquad\qquad or \qquad\qquad r(r-1) + p_0 r + q_0 = 0$$

is known as the **indicial equation** of the given DE. It has tow roots, say r_1, r_2. These roots are called the **exponents at the singularity**. So only for the values $r = r_1$ and $r = r_2$ of r, we can expect to find solutions of (4) in the form of $(*)$.

Is there a chance that $F(r+n)$ may become zero?

If r_1, r_2 are both real and $r_1 \geq r_2$, then clearly $r_1 + n \neq r_2$ for any $n \geq 1$. Hence $F(r_1 + n) \neq 0 \; \forall n \geq 1$. Hence using equation $(**)$, we can find the follows of a_n in terms of r_1 and all the preceeding coefficients $a_0, a_1, \ldots, a_{n-1}$. So we can always hope to determine at least one solution of (4) in the form of $(*)$, namely,

$$y_1(x) = \Sigma_{n=0}^{\infty} a_n x^{n+r_1}$$

And in fact, it has been proved by **Frobenius** that it is possible!

Frobenius Theorem

We can now state the following theorem (due to Frobenius):

THEOREM 6.2 **Frobenius' Theorem**

If $x = x_0$ is a regular singular point of equation (1), then there exists at least one series solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \qquad \textbf{(3)}$$

where the number r is a constant that must be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Remark: In the statement of this theorem, in the last line, we are considering the convergence of the series without the x^r term.
Also if r_1, r_2 are both complex numbers, then they must be conjugate. Hence it is also never possible that $r_1 + n = r_2$ for some positive integer n. Therefore, $F(r_1 + n) \neq 0 \; \forall n \geq 1$. In this case also, we have at least one solution of (4) in the form $(*)$ (Follows from Frobenius theorem):

$$y_1(x) = \Sigma_{n=0}^{\infty} a_n x^{n+r_1} \qquad (a)$$

Another solution exists in some cases!

If r_2 is not equal to r_1 and $r_1 - r_2$ is not a positive integer, then $r_2 + n$ is not equal to r_1 for any value of $n \geq 1$. Hence $F(r_2 + n) \neq 0$, and we can also obtain a second solution

$$y_2(x) = \Sigma_{n=0}^{\infty} a_n x^{n+r_2} \qquad (b)$$

In particular, if r_1 and r_2 are complex numbers, then they are necessarily complex conjugates and $r_2 \neq r_1 + N$. Thus, in this case, we can always find two series solutions of the form (a) and (b), however, they are complex-valued functions of x. Real-valued solutions can be obtained by taking the real and imaginary parts of the complex-valued functions.

Cases of Indicial roots

CASE I: Roots Not Differing by an Integer If r_1 and r_2 are distinct and do not differ by an integer, then there exist two linearly independent solutions of equation (1) of the form

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0 \tag{14a}$$

$$y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0. \tag{14b}$$

Cases of Indicial roots

CASE II: Roots Differing by a Positive Integer If $r_1 - r_2 = N$, where N is a positive integer, then there exist two linearly independent solutions of equation (1) of the form

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0 \tag{20a}$$

$$y_2 = Cy_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0, \tag{20b}$$

where C is a constant that could be zero.

CASE III: Equal Indicial Roots If $r_1 = r_2$, there always exist two linearly independent solutions of equation (1) of the form

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0 \tag{21a}$$

$$y_2 = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}. \tag{21b}$$

Example: Case I

Solve $2xy'' + (1 + x)y' + y = 0.$ **(15)**

SOLUTION If $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, then

$$2xy'' + (1 + x)y' + y = 2\sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n + r)c_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty}(n + r)c_n x^{n+r} + \sum_{n=0}^{\infty}c_n x^{n+r}$$

Example : Case I contd ...

$$= \sum_{n=0}^{\infty} (n+r)(2n+2r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r+1)c_n x^{n+r}$$

$$= x^r \Big[r(2r-1)c_0 x^{-1} + \underbrace{\sum_{n=1}^{\infty} (n+r)(2n+2r-1)c_n x^{n-1}}_{k\,=\,n-1} + \underbrace{\sum_{n=0}^{\infty} (n+r+1)c_n x^n}_{k\,=\,n} \Big]$$

$$= x^r \Big[r(2r-1)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k] x^k \Big] = 0,$$

which implies $\qquad r(2r-1) = 0 \qquad$ **(16)**

$$(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k = 0, \quad k = 0, 1, 2, \ldots . \qquad \textbf{(17)}$$

From (16) we see that the indicial roots are $r_1 = \frac{1}{2}$ and $r_2 = 0$. Because the difference of the indicial roots $r_1 - r_2$ is not an integer, we are guaranteed, as indicated in (14a) and (14b), two linearly independent solutions of the form $y_1 = \sum_{n=0}^{\infty} c_n x^{n+1/2}$ and $y_2 = \sum_{n=0}^{\infty} c_n x^n$.

For $r_1 = \frac{1}{2}$, we can divide by $k + \frac{3}{2}$ in (17) to obtain

$$c_{k+1} = \frac{-c_k}{2(k+1)}$$

$$c_1 = \frac{-c_0}{2 \cdot 1}$$

Example : Case I contd ...

$$c_2 = \frac{-c_1}{2 \cdot 2} = \frac{c_0}{2^2 \cdot 2!}$$

$$c_3 = \frac{-c_2}{2 \cdot 3} = \frac{-c_0}{2^3 \cdot 3!}$$

$$\vdots$$

$$c_n = \frac{(-1)^n c_0}{2^n n!}, \quad n = 1, 2, 3, \ldots .$$

Thus $$y_1 = c_0 x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} x^n\right] = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{n+1/2}, \qquad \textbf{(18)}$$

which converges for $x \geq 0$. As given, the series is not meaningful for $x < 0$ because of the presence of $x^{1/2}$.

Now for $r_2 = 0$, (17) becomes

$$c_{k+1} = \frac{-c_k}{2k+1}$$

$$c_1 = \frac{-c_0}{1}$$

$$c_2 = \frac{-c_1}{3} = \frac{c_0}{1 \cdot 3}$$

Example : Case I contd ...

$$c_3 = \frac{-c_2}{5} = \frac{-c_0}{1 \cdot 3 \cdot 5}$$

$$c_4 = \frac{-c_3}{7} = \frac{c_0}{1 \cdot 3 \cdot 5 \cdot 7}$$

$$\vdots$$

$$c_n = \frac{(-1)^n c_0}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}, \quad n = 1, 2, 3, \ldots .$$

Example : Case I contd ...

We conclude that a second solution of (15) is

$$y_2 = c_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)} x^n \right], \quad |x| < \infty. \qquad \textbf{(19)}$$

On the interval $(0, \infty)$, the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$. ■

Example: Case II

Solve $xy'' + (x - 6)y' - 3y = 0.$ **(22)**

SOLUTION The assumption $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ leads to

$$\begin{aligned} xy'' + (x-6)y' - 3y &= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} - 6\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - 3\sum_{n=0}^{\infty} c_n x^{n+r} \\ &= x^r\Bigg[r(r-7)c_0 x^{-1} + \underbrace{\sum_{n=1}^{\infty}(n+r)(n+r-7)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty}(n+r-3)c_n x^n}_{k=n}\Bigg] \end{aligned}$$

Example: Case II contd ...

$$= x^r \left[r(r-7)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+r+1)(k+r-6)c_{k+1} + (k+r-3)c_k] x^k \right] = 0.$$

Thus $r(r-7) = 0$ so that $r_1 = 7$, $r_2 = 0$, $r_1 - r_2 = 7$, and

$$(k+r+1)(k+r-6)c_{k+1} + (k+r-3)c_k = 0, \quad k = 0, 1, 2, \ldots$$

First find a series solution for $r_1 = 7$ by the usual method of Frobenius. Then for $r_2 = 0$, proceed as follows: PLEASE LOOK AT THE BOARD.

Another way to find a second solution

If we fail to find a second series type solution by the usual method, we can always use the fact that

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

is also a solution of the equation $y'' + P(x)y' + Q(x)y = 0$ whenever y_1 is a known solution.

The above method is certainly needed when $r_1 = r_2$, that is, in case III. It may also be needed sometimes in Case II, when the usual method of Frobenius fails to find the second linearly independent solution corresponding to r_2. For this, see the example below.

Example: Other method needed

EXAMPLE 4 **Only One Series Solution**

Solve $xy'' + y = 0$.

SOLUTION From $xP(x) = 0$, $x^2Q(x) = x$ and the fact that 0 and x are their own power series centered at 0 we conclude that $a_0 = 0$ and $b_0 = 0$, so from (14) the indicial equation is $r(r - 1) = 0$. You should verify that the two recurrence relations corresponding to the indicial roots $r_1 = 1$ and $r_2 = 0$ yield exactly the same set of coefficients. In other words, in this case the method of Frobenius produces only a single series solution

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1} = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots.$$

Example contd ...

$$y_2(x) = y_1(x)\int \frac{e^{-\int 0\,dx}}{[y_1(x)]^2}\,dx = y_1(x)\int \frac{dx}{\left[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots\right]^2}$$

$$= y_1(x)\int \frac{dx}{\left[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots\right]} \qquad \leftarrow \text{after squaring}$$

$$= y_1(x)\int \left[\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots\right]dx \qquad \leftarrow \text{after long division}$$

$$= y_1(x)\left[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots\right] \qquad \leftarrow \text{after integrating}$$

$$= y_1(x)\ln x + y_1(x)\left[-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots\right],$$

or $$y_2(x) = y_1(x)\ln x + \left[-1 - \frac{1}{2}x + \frac{1}{2}x^2 + \cdots\right]. \qquad \leftarrow \text{after multiplying out}$$

On the interval $(0, \infty)$ the general solution is $y = C_1y_1(x) + C_2y_2(x)$. ■