MA 102 (Multivariable Calculus)

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Outline of the Course

Two Topics:

- **Multivariable Calculus**
  Will be taught as the first part of the course. Total Number of Lectures= 21 and Tutorials = 5.

  - G. B. Thomas, Jr. and R. L. Finney, Calculus and Analytic Geometry, 6th/ 9th Edition, Narosa/ Pearson Education India, 1996.
  - T. M. Apostol, Calculus - Vol.2, 2nd Edition, Wiley India, 2003.
  - S. R. Ghorpade and B. V. Limaye, A Course in Multivariable Calculus and Analysis, 1st Indian Reprint, Springer, 2010.

- **Ordinary Differential Equations**
  Will be taught as the second part of the course.

Outline of the Course

Instructors (Multivariable calculus):
Dr. Shreemayee Bora and Dr. Rupam Barman

Course webpage (Calculus): http://www.iitg.ernet.in/rupam/

- For Lecture Divisions and Tutorial Groups, Lecture Venues, Tutorial Venues and Class & Exam Time Tables, See Intranet Academic Section Website.
- Tutorial problem sheets will be uploaded in the course webpage.
  You are expected to try all the problems in the problem sheet before coming to the tutorial class.
  Do not expect the tutor to solve completely all the problems given in the tutorial sheet.

Outline of the Course

Attendance Policy

Attendance in all lecture and tutorial classes is compulsory.

Students, who do not meet 75% attendance requirement in the course, will NOT be allowed to write the end semester examination and will be awarded F (Fail) grade in the course.

(Refer: B.Tech. Ordinance Clause 4.1)

Outline of the Course

Marks distribution:

1. Quiz: 20 percentage

   (Two quizes: Quiz-1 from multivariable calculus and Quiz-2 from ODE)
2. Mid-term: 30 percentage
3. End-term: 50 percentage (20% will be on multivariable calculus)

No make up test for Quizzes and Mid Semester Examination.

Do preserve your (evaluated) answer scripts of Quizzes and Mid Semester Examination of MA102 till the completion of the Course Grading.

Introduction

The aim of studying the functions depending on several variables is to understand the functions which has several input variables and one or more output variables.

For example, the following are real valued functions of two variables x, y:

(1) $f(x,y) = x^2 + y^2$ is a real valued function defined over $\mathbb{R}^2$.

(2) $f(x,y) = \frac{xy}{x^2+y^2}$ is a real valued function defined over $\mathbb{R}^2 \setminus \{(0,0)\}$

Optimal cost functions: For example, a manufacturing company wants to optimize the resources like man power, capital expenditure, raw materials etc. The cost function depends on these variables.

Earning per share for Apple company (2005-2010) has been modeled by $z = 0.379x - 0.135y - 3.45$ where x is the sales and y is the share holders equity.

Example

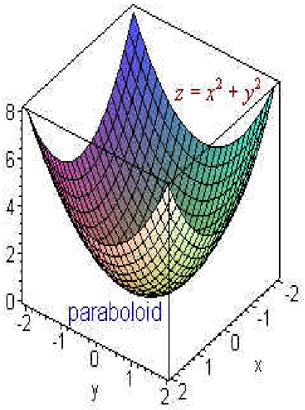


Figure: Paraboloid $z = f(x, y) = x^2 + y^2$, f is a function from $\mathbb{R}^2$ to $\mathbb{R}$

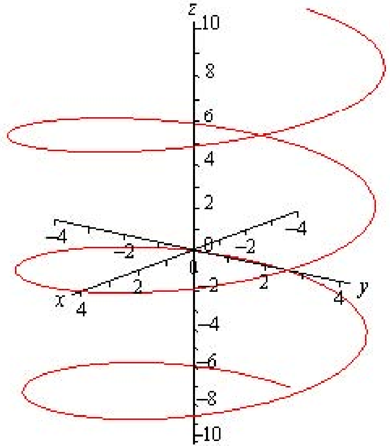


Figure: Helix $\mathbf{r}(t) = (4\cos t, 4\sin t, t)$, $\mathbf{r}$ is a function from $\mathbb{R}$ to $\mathbb{R}^3$

Review of Analysis in $\mathbb{R}$

- $(\mathbb{R}, +, \cdot)$ is an ordered field.
- Completeness property.
- Monotone convergence property:
  Bounded + Monotone $\Rightarrow$ Convergent
- $(\mathbb{R}, |\cdot|)$ is complete
- Bolzano-Weierstrass Thm: A bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Euclidean space $\mathbb{R}^n$

Euclidean space $\mathbb{R}^n$:

- $\mathbb{R}^n := \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$.
- If $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, then $X + Y := (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and $\alpha \cdot X := (\alpha x_1, \alpha x_2, \ldots, \alpha x_n), \alpha \in \mathbb{R}$.
- $(\mathbb{R}^n, +, \cdot)$ is a vector space over $\mathbb{R}$.

Euclidean norm in $\mathbb{R}^n$: For $X \in \mathbb{R}^n$, we define

$$\|X\| := (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}.$$

Euclidean space $\mathbb{R}^n$

Fundamental properties of Euclidean norm:

(i) $\|X\| \geq 0$ and $\|X\| = 0$ if and only if $X = 0$.

(ii) $\|\alpha \cdot X\| = |\alpha|\|X\|$ for every $\alpha \in \mathbb{R}$ and $X \in \mathbb{R}^n$.

(iii) $\|X + Y\| \leq \|X\| + \|Y\|$ for all $X, Y \in \mathbb{R}^n$.

Euclidean distance in $\mathbb{R}^n$: For $X, Y \in \mathbb{R}^n$, the Euclidean distance between X and Y is defined as

$$d(X, Y) := \|X - Y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Inner product/ dot product

Inner product/ dot product: $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
$\langle X, Y \rangle = x_1 y_1 + \cdots + x_n y_n = X \bullet Y$.

We have $\|X\| = \sqrt{\langle X, X \rangle}$

Let θ be the angle between two nonzero vectors X and Y

Then,

$$\cos\theta = \frac{\langle X, \ Y \rangle}{||X|| \, ||Y||} .$$

Orthogonality: If $\langle X, Y \rangle = 0$, then $X \perp Y$.

Inner product/ dot product

Cauchy-Schwarz Inequality:

$$|\langle X, Y\rangle| \leq \|X\|\|Y\|$$

Parallelogram Law:

$$\|X\|^2+\|Y\|^2 = \frac{1}{2}\left(\|X+Y\|^2 + \|X-Y\|^2\right) \text{ for all } X,\ Y \in \mathbb{R}^n .$$

Polarization Identity:

$$\langle X,\ Y\rangle = \frac{1}{4}\left(\|X+Y\|^2 - \|X-Y\|^2\right) \text{ for all } X,\ Y \in \mathbb{R}^n .$$

Which can go wrong in higher dimensional situation?

Let $\{a_{ij} \in \mathbb{R} \ : \ 1 \leq i \leq m, \ 1 \leq j \leq n\}$ be a two-dimensional array.

Then $\displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}$ holds.

- Let $\{a_{ij} \in \mathbb{R} \ : \ i \in \mathbb{N}, \ j \in \mathbb{N}\}$.

  Does $\displaystyle\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ hold?

  Let a_{ij} be defined as

$$a_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -1 & \text{if } i = j+1; \\ 0 & \text{otherwise.} \end{cases}$$

We arrange the numbers a_{ij} in column and rows

$$\begin{array}{ccccc} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Then

$$\sum_i \sum_j a_{ij} = \text{row-sum} = 1 + 0 + 0 + \cdots = 1$$

$$\sum_j \sum_i a_{ij} = \text{column-sum} = 0 + 0 + 0 + \cdots = 0$$

Let a_{ij} be defined as

$$a_{ij} = \begin{cases} 0 & \text{if } j > i; \\ -1 & \text{if } i = j; \\ 2^{j-i} & \text{if } i > j. \end{cases}$$

Then

$$\sum_i \sum_j a_{ij} = \text{row-sum} = -2$$

$$\sum_j \sum_i a_{ij} = \text{column-sum} = 0$$

Which can go wrong in higher dimensional situation?

- Let $f(x, y) = \dfrac{x^2}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$.

$$\lim_{x \to 0} \left(\lim_{y \to 0} f(x, y) \right) = \lim_{x \to 0} 1 = 1 .$$

$$\lim_{y \to 0} \left(\lim_{x \to 0} f(x, y) \right) = \lim_{y \to 0} 0 = 0 .$$

- Let $f(x, y) = \dfrac{x^2 y^2}{x^2 y^2 + (x - y)^2}$ if $x^2 y^2 + (x - y)^2 \neq 0$

$$\lim_{x \to 0} \left(\lim_{y \to 0} f(x, y) \right) = \lim_{x \to 0} 0 = 0 .$$

$$\lim_{y \to 0} \left(\lim_{x \to 0} f(x, y) \right) = \lim_{y \to 0} 0 = 0 .$$

Which can go wrong in higher dimensional situation?

Let $f(x,\ y) = e^{-xy} - xy\, e^{-xy}$.
Compute the iterated integral of f as x varies from 0 to ∞ and y varies from 0 to 1.

$$\int_{x=0}^{\infty} \left(\int_{y=0}^{1} f(x,\ y)\, dy \right) dx = \int_{x=0}^{\infty} \left[ye^{-xy} \right]_{y=0}^{1} dx = \int_{x=0}^{\infty} e^{-x}\, dx = 1\ .$$

$$\int_{y=0}^{1} \left(\int_{x=0}^{\infty} f(x,\ y)\, dx \right) dy = \int_{y=0}^{1} \left[xe^{-xy} \right]_{x=0}^{\infty} dy = \int_{y=0}^{1} 0\, dy = 0\ .$$

That is,

$$\int_{x=0}^{\infty} \left(\int_{y=0}^{1} f(x,\ y)\, dy \right) dx = 1 \neq 0 = \int_{y=0}^{1} \left(\int_{x=0}^{\infty} f(x,\ y)\, dx \right) dy\ .$$

Which can go wrong in higher dimensional situation?

Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where $n > 1$ and $m > 1$.
How to define differentiability of F?
Let $X_0 \in D$.

$$\frac{F(X) - F(X_0)}{X - X_0}$$

Numerator is $(F(X) - F(X_0)) \in \mathbb{R}^m$ which is a vector quantity.
Denominator is $(X - X_0) \in \mathbb{R}^n$ which is a vector quantity.

We are unable to define the quantity $\frac{F(X)-F(X_0)}{X-X_0}$.

So, How to overcome this difficulty in order to define the differentiability of F?

What are the things same in higher dimensional situation?

- Concept of Convergence of Sequences is same.
- Concept of Limits of Functions is same.
- Concept of Continuity of Functions is same.

- Differentiation can not be taken as such to the higher dimension.
- Integration can not be taken as such to the higher dimension.
- Differentiation and Integration can be taken as such to the functions $F : (a,\ b) \subset \mathbb{R} \to \mathbb{R}^n$ where $n > 1$.
- Riemann Integration can be taken as such to the functions $f : D \subset \mathbb{R}^n \to \mathbb{R}$ where $n > 1$ and $D = \{(x_1,\ \cdots, x_n) \in \mathbb{R}^n\ :\ x_i \in [a_i,\ b_i],\ 1 \le i \le n\}$.

Notations

- We denote Vectors by writing in the Capital Letters like X, V, Z, etc.
  Usually, in the books, vectors are denoted by the bold face letters like **x**, **v**, etc.
- We denote Scalars by writing in the Small Letters like x, s, a, λ, etc.
- In $\mathbb{R}^n$, we usually take the Euclidean norm. We use $\| \cdot \|$ interchangeably with $| \cdot |$.
- In $\mathbb{R}^n$, we usually take the vectors dot product as an innerproduct. We use $\langle X, Y \rangle$ interchangeably with $X \cdot Y$.

Convergence of sequence in $\mathbb{R}^n$

Definition 1: A function $\mathbb{N} \to \mathbb{R}^n$, $k \mapsto X_k$ is called a sequence.

Note that each term X_k is a vector in $\mathbb{R}^n$. That is,

$$X_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n}).$$

Thus, given a sequence $\langle X_k \rangle_{k=1}^{\infty}$ in $\mathbb{R}^n$, we obtain n sequences in $\mathbb{R}$, namely, $\langle x_{k,1} \rangle_{k=1}^{\infty}$, $\langle x_{k,2} \rangle_{k=1}^{\infty}, \ldots$, $\langle x_{k,n} \rangle_{k=1}^{\infty}$.

Definition 2: Let $X_k, X \in \mathbb{R}^n$. The sequence $\langle X_k \rangle$ is said to converge to X if $d(X_k, X) = \|X_k - X\| \to 0$ as $k \to \infty$. That is, for given $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that $\|X_k - X\| < \varepsilon$ whenever $k \geq p$.

Convergence of sequence in $\mathbb{R}^n$

Theorem: Let $X_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n}) \in \mathbb{R}^n$ and $X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then $X_k \to X$ if and only if $x_{k,j} \to x_j$ for each $j = 1, 2, \ldots, n$.

Moral: Convergence of sequence in $\mathbb{R}^n$ is essentially same as that in $\mathbb{R}$.

Convergence of sequence in $\mathbb{R}^n$

Definition: A sequence $\langle X_k \rangle$ in $\mathbb{R}^n$ is said to be Cauchy if $d(X_k, X_\ell) = \|X_k - X_\ell\| \to 0$ as $k, \ell \to \infty$. That is, for given $\varepsilon > 0$, there exists $p \in \mathrm{N}$ such that $\|X_k - X_\ell\| < \varepsilon$ whenever $k, \ell \geq p$.

Theorem: $\mathbb{R}^n$ is complete. That is, every Cauchy sequence in $\mathbb{R}^n$ is convergent.

Convergence of sequence in $\mathbb{R}^n$

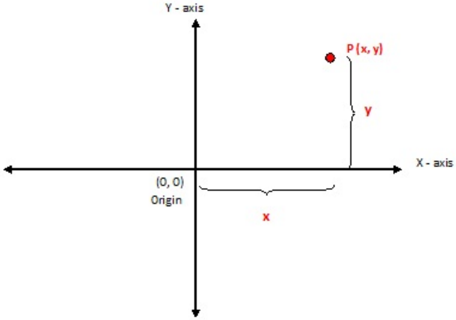
A subset A of $\mathbb{R}^n$ is called bounded if there exists a constant $K > 0$ such that $\|X\| \leq K$ for all $X \in A$.

Bolzano-Weierstrass Theorem in $\mathbb{R}^n$: If $\langle X_k \rangle$ is bounded in $\mathbb{R}^n$, then it has a convergent subsequence.

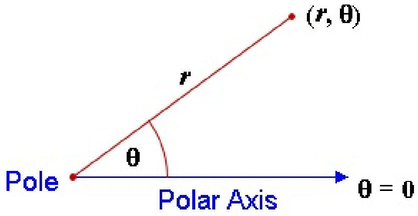
Proof: Given a sequence $\langle X_k \rangle_{k=1}^{\infty}$ in $\mathbb{R}^n$, we obtain n sequences in $\mathbb{R}$, namely, $\langle x_{k,1} \rangle_{k=1}^{\infty}$, $\langle x_{k,2} \rangle_{k=1}^{\infty}, \ldots,$ $\langle x_{k,n} \rangle_{k=1}^{\infty}$. We now apply Bolzano-Weierstrass Theorem to each of these n sequences in $\mathbb{R}$.

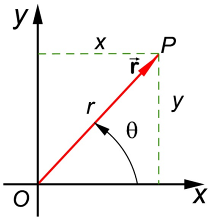
$\mathbb{R}^2$: Cartesian Coordinates/ Rectangular Coordinates

Any point P in the plane (in 2-D) can be assigned coordinates in the rectangular (or cartesian) coordinates system as (x, y).



$\mathbb{R}^2$: Polar Coordinates



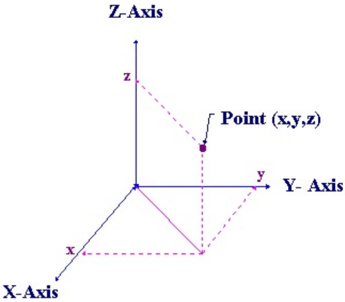


For each nonzero point $P = (x,\ y) \neq (0,\ 0)$, the polar coordinates $(r,\ \theta)$ of P are given by the equations

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ \text{or}\ \ \ x^2 + y^2 = r^2,\ \frac{y}{x} = \tan\theta\ .$$

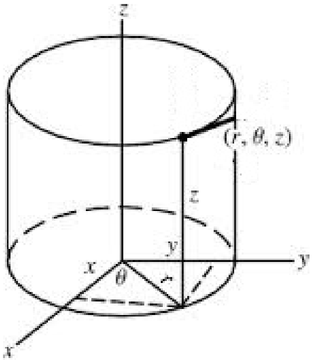
The point $(r,\ \theta)$ and $(r,\ \theta + 2n\pi)$ where n is any integer denote the same (geometrical) point.

$\mathbb{R}^3$: Cartesian Coordinates/ Rectangular Coordinates



$\mathbb{R}^3$: Cylindrical Coordinates

A cylindrical coordinate system consists of polar coordinates $(r,\ \theta)$ in a plane together with a third coordinate z measured along an axis perpendicular to the $r\theta$-plane which is the xy-plane. This means that the z-coordinate in the cylindrical coordinate system is the same as the z-coordinate in the cartesian system.



Relation between Cylindrical and Rectangular Coordinates

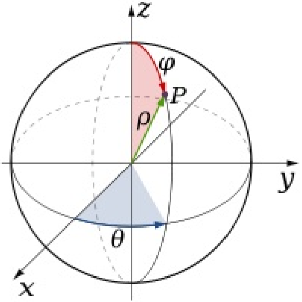
Cylindrical and Rectangular coordinates are related by the following equations:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

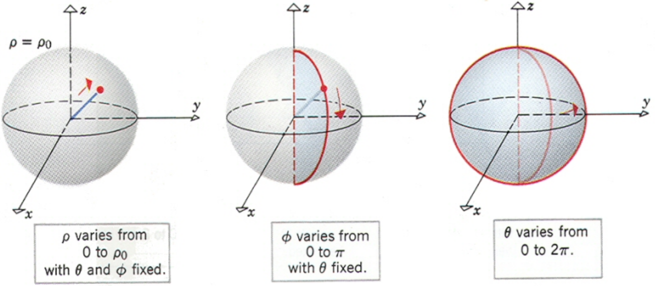
where $r^2 = x^2 + y^2$ *and* $\tan\theta = (y/x)$.

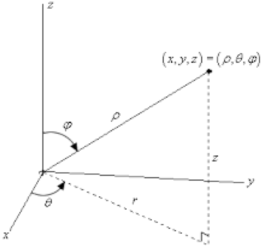
$\mathbb{R}^3$: Spherical Coordinates

Spherical coordinates are useful when there is a center of symmetry that we can take as the origin. The spherical coordinates $(\rho, \ \phi, \ \theta)$ of a given point A are shown in the following Figure



Here $\rho = \sqrt{x^2 + y^2 + z^2}$.


$\rho = \rho_0$
ρ varies from 0 to ρ_0 with θ and ϕ fixed.
ϕ varies from 0 to π with θ fixed.
θ varies from 0 to 2π.


$(x, y, z) = (\rho, \theta, \varphi)$
ρ
φ
θ
r
z
x
y

The relation between spherical coordinates and cylindrical coordinates are given by

$$\begin{aligned} r &= \rho \sin \phi \\ z &= \rho \cos \phi \\ \theta &= \theta \end{aligned}$$

The relation between spherical coordinates and cartesian coordinates are given by

$$\begin{aligned} x &= \rho \sin \phi \ \cos \theta \\ y &= \rho \sin \phi \ \sin \theta \\ z &= \rho \cos \phi \end{aligned}$$

Topology on $\mathbb{R}^n$

Open Ball: Let $\varepsilon > 0$ and $X_0 \in \mathbb{R}^n$. Then

$$B(X_0, \varepsilon) := \{X \in \mathbb{R}^n : \|X - X_0\| < \varepsilon\}$$

is called open ball of radius ε centred at X_0.

Topology on $\mathbb{R}^n$

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$$B(X_0, \varepsilon) := \{X \in \mathbb{R}^n : \|X - X_0\| < \varepsilon\}$$

is called open ball of radius ε centred at X_0.

Let S be a subset of $\mathbb{R}^n$.
Interior point: A point X_0 is said to be an interior point of S if there is some $\varepsilon > 0$ such that $B(X_0, \varepsilon) \subseteq S$.

Open set: $O \subset \mathbb{R}^n$ is open if for any $X \in O$ there is $\varepsilon > 0$ such that $B(X, \varepsilon) \subset O$.
That is, every point of O is an interior point.

Topology on $\mathbb{R}^n$

Examples:

1. $B(X, \varepsilon) \subset \mathbb{R}^n$ is an open set.
2. $O := (a_1, b_1) \times \cdots \times (a_n, b_n)$ is open in $\mathbb{R}^n$.
3. $\mathbb{R}^n$ is open.
4. Union of open balls is an open set.

Facts:

1. The interior of a set is always an open set.
2. The interior of a set S is the largest open set contained in the set S.
3. S is open if and only if S is equal to its interior.

Closed set: $S \subset \mathbb{R}^n$ is closed if $S^c := \mathbb{R}^n \setminus S$ is open.

Examples:

1. $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ is a closed set.
2. $C(X_0, \varepsilon) := \{X \in \mathbb{R}^n : \|X - X_0\| \leq \varepsilon\}$ is a closed set.
3. $E := [a_1, b_1] \times \cdots \times [a_n, b_n]$ is closed in $\mathbb{R}^n$.
4. $\mathbb{R}^n$ is closed.

Closed set: $S \subset \mathbb{R}^n$ is closed if $S^c := \mathbb{R}^n \setminus S$ is open.

Examples:

1. $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ is a closed set.
2. $C(X_0, \varepsilon) := \{X \in \mathbb{R}^n : \|X - X_0\| \leq \varepsilon\}$ is a closed set.
3. $E := [a_1, b_1] \times \cdots \times [a_n, b_n]$ is closed in $\mathbb{R}^n$.
4. $\mathbb{R}^n$ is closed.

Theorem: Let $S \subset \mathbb{R}^n$. Then the following are equivalent:

1. S is closed.
2. If $(X_k) \subset S$ and $X_k \to X \in \mathbb{R}^n$ then $X \in S$.

Limit point: Let $A \subset \mathbb{R}^n$ and $X \in \mathbb{R}^n$. Then X is a limit point of A if $A \cap (B(X, \varepsilon) \setminus \{X\}) \neq \emptyset$ for any $\varepsilon > 0$.

Examples:

1. Each point in $B(X, \varepsilon)$ is a limit point.
2. Each $Y \in \mathbb{R}^n$ such that $\|X - Y\| = \varepsilon$ is a limit point of $B(X, \varepsilon)$.

Limit point: Let $A \subset \mathbb{R}^n$ and $X \in \mathbb{R}^n$. Then X is a limit point of A if $A \cap (B(X, \varepsilon) \setminus \{X\}) \neq \emptyset$ for any $\varepsilon > 0$.

Examples:

1. Each point in $B(X, \varepsilon)$ is a limit point.
2. Each $Y \in \mathbb{R}^n$ such that $\|X - Y\| = \varepsilon$ is a limit point of $B(X, \varepsilon)$.

Theorem: Let $S \subset \mathbb{R}^n$. Then S is closed $\iff$ S contains all of its limit points.

Closure of a Set

Let S be a subset of $\mathbb{R}^n$.

The set S together with all its limit points is called the closure of a set and is denoted by $\overline{S}$ or $\mathrm{Cl}(S)$.

- The closure of a set is always a closed set.
- The closure of a set S is the smallest closed set containing the set S.
- S is closed if and only if $S = \overline{S}$.
- Empty set $\emptyset$ and the whole set $\mathbb{R}^n$ are both open and closed sets.

Boundary Point, Exterior Point

Let S be a subset of $\mathbb{R}^n$.

A point X_0 is said to be a boundary point of S if every open ball $B(X_0, \varepsilon)$ centered at X_0 contains points from S as well as points from the complement of S.

A point X_0 is said to be an exterior point of S if there is some $\varepsilon > 0$ such that $B(X_0, \varepsilon) \subseteq S^c$, where S^c is the complement of S.

That is, X_0 is the interior point of S^c.

Limit of a function

Definition:

- Let $f : \mathbb{R}^n \to \mathbb{R}$, $X_0 \in \mathbb{R}^n$ and $L \in \mathbb{R}$. Then $\lim_{X \to X_0} f(X) = L$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$0 < \|X - X_0\| < \delta \Longrightarrow |f(X) - L| < \varepsilon.$$

Limit of a function

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$$0 < \|X - X_0\| < \delta \Longrightarrow |f(X) - L| < \varepsilon.$$

- Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $L \in \mathbb{R}$. Let $X_0 \in \mathbb{R}^n$ be a limit point of A. Then $\lim_{X \to X_0} f(X) = L$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$X \in A \text{ and } 0 < \|X - X_0\| < \delta \Longrightarrow |f(X) - L| < \varepsilon.$$

Limit of a function

Example 1: Consider the function f defined by

$$f(x,y) = \frac{4xy^2}{x^2+y^2}.$$

This function is defined in $\mathbb{R}^2 \setminus \{(0,0)\}$. Let $\varepsilon > 0$. Since $4|xy^2| \leq 4\sqrt{x^2+y^2}(x^2+y^2)$, for $(x,y) \neq (0,0)$, we have

$$|f(x,y) - 0| = \left|\frac{4xy^2}{x^2+y^2}\right| \leq 4\sqrt{x^2+y^2} < \varepsilon,$$

whenever $\|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$, where $\delta = \varepsilon/4$.

Hence, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Limit of a function

Example 2 (Finding limit through polar coordinates):

Consider the function $f(x, y) = \dfrac{x^3}{x^2 + y^2}$. This function is defined in $\mathbb{R}^2 \setminus \{(0, 0)\}$. Taking $x = r\cos\theta, y = r\sin\theta$, we get

$$|f(r, \theta)| = |r\cos^3\theta| \leq r \to 0 \ \ as \ \ r \to 0.$$

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$$|f(r, \theta)| = |r\cos^3\theta| \leq r \to 0 \ \ as \ \ r \to 0.$$

Hence, $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$.

Remark: Note that $(0, 0)$ is a limit point of $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Sequential characterization

Theorem: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$, $L \in \mathbb{R}$ and $X_0 \in \mathbb{R}^n$ be a limit point of A. Then the following are equivalent:

- $\lim_{X \to X_0} f(X) = L$
- If $(X_k) \subset A \setminus \{X_0\}$ and $X_k \to X_0$ then $f(X_k) \to L$.

Proof: Exercise.

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Proof: Exercise.

Remark:

- Limit, when exists, is unique.
- Sum, product and quotient rules hold.

Examples:

1. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) := 0$ and $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

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2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) := \begin{cases} x\sin(1/y) + y\sin(1/x) & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

   Then $\lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Iterated limits

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. Then $\lim_{x \to a} \lim_{y \to b} f(x, y)$, when exists, is called an iterated limit of f at (a, b).

Similarly, one defines $\lim_{y \to b} \lim_{x \to a} f(x, y)$.

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Similarly, one defines $\lim_{y\to b} \lim_{x\to a} f(x, y)$.

Remark:

- Iterated limits are defined similarly for $f : A \subset \mathbb{R}^n \to \mathbb{R}$.
- Existence of limit does not guarantee existence of iterated limits and vice-versa.
- Iterated limits when exist may be unequal. However, if limit and iterated limits exist then they are all equal.

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Then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

However,

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0 = \lim_{y\to 0}\lim_{x\to 0} f(x,y).$$

Example 2: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) := \begin{cases} x\sin(1/y) + y\sin(1/x) & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

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Both the iterated limits do not exist.

Example 3: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) := 0$ and $f(x,y) := \frac{x^2-y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

Example 3: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) := 0$ and $f(x,y) := \frac{x^2-y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

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We have $\lim_{x\to 0} \lim_{y\to 0} f(x,y) = 1$ and $\lim_{y\to 0} \lim_{x\to 0} f(x,y) = -1$.

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Note that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.