Indian Institute of Technology Guwahati

MA 102 Mathematics-II
Tutorial Sheet-5

Line, surface integrals and Stoke's, divergence theorems

Section-I: Tutorial Problems

1. What is the integral of the function x^2z taken over the entire surface of a right circular cylinder of height h which stands on the circle $x^2 + y^2 = a^2$. What is the integral of the given function taken throughout the volume of the cylinder.

2. Find the line integral of the vector field $F(x, y, z) = y\vec{i} - x\vec{j} + \vec{k}$ along the path $c(t) = (\cos t, \sin t, \frac{t}{2\pi})$, $0 \leq t \leq 2\pi$ joining $(1, 0, 0)$ to $(1, 0, 1)$.

3. Evaluate $\int_C T \cdot dR$, where C is the circle $x^2 + y^2 = 1$ and T is the unit tangent vector.

4. Show that the integral $\int_C yzdx + (xz + 1)dy + xydz$ is independent of the path C joining $(1, 0, 0)$ and $(2, 1, 4)$.

5. Use Green's Theorem to compute $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where C is the boundary of the region $\{(x, y) : x, y \geq 0 \ \& \ x^2 + y^2 \leq 1\}$.

6. Use Stokes' Theorem to evaluate the line integral $\int_C -y^3dx + x^3dy - z^3dz$, where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.

7. Let $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and let S be any surface that surrounds the origin. Prove that $\iint_S \vec{F} \cdot n d\sigma = 4\pi$.

8. Let D be the domain inside the cylinder $x^2 + y^2 = 1$ cut off by the planes $z = 0$ and $z = x + 2$. If $\vec{F} = (x^2 + ye^z, y^2 + ze^x, z + xe^y)$, use the divergence theorem to evaluate $\iint_{\partial D} F \cdot \mathbf{n} \, d\sigma$.

Section-II: Practice Problems

1. Consider the surface $z = x^2 + y^2 + 1$.

   (a) Show that $\gamma(r, \theta) = (r\cos\theta, r\sin\theta, r^2 + 1)$, $r \geq 0$, $0 \leq \theta \leq 2\pi$ is a parametrization of the surface.

   (b) Parametrize the surface in the variables z and θ using cylindrical coordinates.

2. Let S be the part of sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cone $z = \sqrt{x^2 + y^2}$. Parametrize S by considering it as graph and again by using spherical coordinates.

3. Let S be the hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 4, \ z \geq 0\}$.

(a) Evaluate $\iint\limits_S z^2 d\sigma$ by considering S as a graph $z = f(x, y)$.

(b) Evaluate $\iint\limits_S z d\sigma$ by considering S as a parametric surface (but not as a graph).

4. Let S be the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant. Evaluate $\iint\limits_S (z + 2xy) d\sigma$

5. Let S denote the part of the plane $2x+5y+z = 10$ that lies inside the cylinder $x^2+y^2 = 9$. Find the area of S.

   (a) By considering S a part of the graph $z = f(x, y)$, where $f(x, y) = 10 - 2x - 5y$.

   (b) By considering S as a parametric surface $\gamma(u, v) = (u \cos v, u \sin v, 10 - u(2 \cos v + 5 \sin v))$, where $0 \leq u \leq 2\pi$.

6. Find the area of the surface $x = uv,\ y = u + v,\ z = u - v$ where $(u, v) \in D = \{(s, t) \in \mathbb{R}^2 : s^2 + t^2 \leq 1\}$.

7. Let $\overrightarrow{N}$ the unit outward normal vector on the ellipse $x^2 + 2y^2 = 1$. Evaluate the line integral $\int\limits_C \overrightarrow{N}.\overrightarrow{dR}$ along the circle $C = \{(x, y) :\ x^2 + y^2 = 1\}$.

8. Use fundamental theorem of calculus for line integral to show that $\int_C y\ dx + (x + z)\ dy + y\ dz$ is independent of any path C joining the points $(2, 1, 4)$ and $(8, 3, -1)$.

9. Find the surface integral $\iint\limits_S zdS$, where S it the part of the paraboloid $2z = x^2 + y^2$ which lies in the cylinder $x^2 + y^2 = 1$.

10. Let C be the boundary of the cone $z = x^2 + y^2$ and $0 \leq z \leq 1$. Use Stoke's theorem to evaluate the line integral $\int_C \overrightarrow{F}.\overrightarrow{dg}$ where $\overrightarrow{F} = (y, xz, 1)$.

11. Let $\overrightarrow{F} = (xy, yz, zx)$ and S be the surface $z = 4 - x^2 - y^2$ with $2 \leq z \leq 4$. Use divergence theorem to find the surface integral $\iint\limits_S \overrightarrow{F}.\overrightarrow{n}\, dS$.