Indian Institute of Technology Guwahati

MA 102 Mathematics-II
Tutorial Sheet-4

Double, triple integrals and change of variables

Section-I: Tutorial Problems

1. Evaluate the following integrals.

   (a) $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dydx$

   (b) $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dydx$

   (c) $\int_0^1 \int_y^1 x^2 \exp^{xy}\,dxdy$

2. Evaluate $\iint_R x dxdy$ where R is the region $1 \le x(1-y) \le 2$ and $1 \le xy \le 2$.

3. Using double integral, find the area enclosed by the curve $r = \sin 3\theta$ given in polar coordinates.

4. Compute $\lim_{a\to\infty} \iint_{D(a)} \exp^{-(x^2+y^2)}\,dxdy$, where

   (a) $D(a) = \{(x, y) : x^2 + y^2 \le a^2\}$ and

   (b) $D(a) = \{(x, y) : 0 \le x \le a, 0 \le y \le a\}$.

   Hence prove that $\int_0^\infty \exp^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

5. Evaluate the integral $\iiint_W \frac{dzdydx}{\sqrt{1+x^2+y^2+z^2}}$, where W is the ball $x^2 + y^2 + z^2 \le 1$.

6. Let D denote the solid bounded by the surfaces $y = x$, $y = x^2$, $z = x$ and $z = 0$. Evaluate $\iiint_D y dxdydz$.

Section-II: Practice Problems

1. Consider the transformation $T : [0, 2\pi] \times [0, 1] \to \mathbb{R}^2$ given by $T(u, v) = (2v \cos u, v \sin u)$.

   (a) For a fixed $v_o \in [0, 1]$, describe the set $\{T(u, v_o) : u \in [0, 2\pi]\}$.

   (b) Describe the set $\{T(u, v) : [0, 2\pi] \times [0, 1]\}$.

2. Let R be the region in $\mathbb{R}^2$ bounded by the straight lines $y = x$, $y = 3x$ and $x + y = 4$. Consider the transformation $T(u, v) = (u - v, u + v)$. Find the set S satisfying $T(S) = R$.

3. Evaluate the following iterated integrals by interchanging the order of integration.

   (a) $\int_0^1 \int_{x=y}^1 \cos(x^2)dxdy$.

   (b) $\int_0^1 \int_{y=\sqrt{x}}^1 e^{y^3}dydx$.

   (c) $\int_0^1 \int_{y=x^2}^1 x^3e^{y^3}dydx$.

   (d) $\int_0^1 \int_{x=y}^1 \frac{1}{1+x^4}dxdy$.

4. Let $R = [a,b] \times [c,d]$ and $f : R \to \mathbb{R}$ be defined by $f(x,y) = p(x)q(y)$, where $p : [a,b] \to \mathbb{R}$ and $q : [c,d] \to \mathbb{R}$ are continuous. Show that $\iint_R f(x,y)dxdy = \left(\int_a^b p(x)dx\right)\left(\int_c^d q(x)dx\right)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $\int_{y=0}^x \int_{t=0}^y f(t)dtdy = \int_{t=0}^x (x-t)f(t)dt$.

6. Let R be the region lying below the curve $y = \cos x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and above the X-axis. Evaluate $\iint_R \sin xdxdy$.

7. Let R be the region in $\mathbb{R}^2$ bounded by the curves $y = 2x^2$ and $y = 1 + x^2$. Evaluate the double integral $\iint_R (2x^2 + y)dxdy$.

8. Evaluate $\iint_R x\cos\left(y - \frac{y^3}{3}\right)dxdy$, where $R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$.

9. Evaluate the double integral $\iint_D \sqrt{x+y}\ (y-2x)^2dydx$ over the domain D bounded by the lines $x = 0,\ \ y = 0$ and $x + y = 1$.

10. Find the volume of the solid enclosed by the surfaces $z = 6-x^2-y^2,\ z = 2x^2+y^2-1,\ x = -1,\ x = 1,\ y = -1$ and $y = 1$.

11. Let D be the solid bounded by the surfaces $y = x^2,\ y = 3x,\ z = 0$ and $z = x^2 + y^2$. Find the volume of the solid.

12. Let D be the solid bounded by the cylinder $x^2 + y^2 = 1$and the planes $y + z = 1$ and $z = 0$. Find the volume of the solid.

13. Using change of variables $u = x + y$ and $v = x - y$, show that

$$\int_0^1 \int_{y=0}^{y=x} (x-y)dydx = \int_0^1 \int_{u=v}^{2-v} \frac{v}{2}dudv.$$

14. Evaluate

(a) $\int\limits_{0}^{\frac{1}{\sqrt{2}}} \int\limits_{x=y}^{\sqrt{1-y^2}} (x+y)dxdy.$

(b) $\int\limits_{0}^{1} \int\limits_{x=0}^{1-y} \sqrt{x+y}(y-2x)^2 dxdy.$

(c) $\int\limits_{1}^{2} \int\limits_{x=0}^{y} \frac{1}{(x^2+y^2)^{\frac{3}{2}}} dxdy.$

(d) $\int\limits_{0}^{2} \int\limits_{y=0}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} dydx.$

15. Find the volume of the solid in the first octant bounded below by the surface $z = \sqrt{x^2+y^2}$ above by $x^2+y^2+z^2 = 8$ as well as the planes $y = 0$ and $y = x$.

16. Let D denote the solid bounded below by the plane $z + y = 2$, above by the cylinder $z + y^2 = 4$ and on the sides $x = 0$ and $x = 2$. Evaluate $\iiint\limits_{D} xdxdydz$.

17. Let $D = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \leq 1\}$ and $E = \{(u,v,w) \in \mathbb{R}^3 : u^2+v^2+w^2 \leq 1\}$. Show that $\iiint\limits_{D} dxdydz = \iiint\limits_{E} 24dudvdw$.

18. Let D be the solid that lies inside the cylinder $x^2+y^2 = 1$, below the cone $z = \sqrt{4(x^2+y^2)}$ and above the plane $z = 0$. Evaluate $\iiint\limits_{D} x^2 dxdydz$.

19. Evaluate $\int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int\limits_{x^2+y^2}^{4} xdzdydx.$

20. Let D denote the solid bounded above by the plane $z = 4$ and below by the cone $z = \sqrt{x^2+y^2}$. Evaluate $\iiint\limits_{D} \sqrt{x^2+y^2+z^2} dxdydz$.

21. Let D denote the solid enclosed by the spheres $x^2+y^2+(z-1)^2 = 1$ and $x^2+y^2+z^2 = 3$. Using the spherical coordinates, set up iterated integral that gives the volume of D.