Indian Institute of Technology Guwahati

MA 102 Mathematics-II
Tutorial Sheet-3

MVT, Maxima and Minima

Section-I: Tutorial Problems

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be function given by $z = f(x,y) = x^4 + y^4$. Find a point on the surface $z = f(x,y)$, where normal to the surface is perpendicular to the chord joining the points $(0,0,f(0,0))$ and $(1,1,f(1,1))$.

2. Examine the following functions for local maxima, local minima and saddle points:

   (a) $f(x,y) = 4xy - x^4 - y^4$

   (b) $f(x,y) = x^3 - 3xy$

3. Find the absolute maxima of $f(x,y) = xy$ on the unit disc $\{(x,y) : x^2 + y^2 \leq 1\}$.

4. Minimize the quantity $x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

Section-II: Practice Problems

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a map defined by $f(x,y) = x^2 + y^2$. Find the maximum rate of change of f along the ellipse $R(t) = (a\cos t, b\sin t),\ t \in [0, 2\pi)$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be function given by $f(x_1, x_2, \ldots, x_n) = \sin(x_1 + x_2 + \cdots, x_n)$. For the points $X,\ Y \in \mathbb{R}^n$, show that $|f(Y) - f(X)| \leq \sqrt{n}\ \|Y - X\|$.

3. Let $f(x,y) = \frac{2x^3y - 3xy^3}{x^2+y^2}$, if $x^2 + y^2 \neq 0$ and $f(0,0) = 0$. Show that, at $(0,0)$,

   (a) f is continuous.

   (b) f_x and f_y are continuous.

   (c) f is differentiable.

   (d) $f_{xy} \neq f_{yx}$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{x^2y(x-y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

   Examine whether $f_{xy}(0,0) = f_{yx}(0,0)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ Determine all the points of $\mathbb{R}^2$ where f_{xy} and f_{yx} are continuous.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Show that f is convex if and only if $f(Y) \geq f(X) + f'(X).(Y - X)$, for all $X, Y \in \mathbb{R}^2$.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that f_x and f_y are continuous and they have continuous partial derivatives. Then f is convex if, for all $X \in \mathbb{R}^2$, the matrix $M_X = \begin{pmatrix} f_{xx}(X) & f_{xy}(X) \\ f_{yx}(X) & f_{yy}(X) \end{pmatrix}$ is non-negative definite.

8. Let $f(x,y) = (x-y)(x-y^2)$. Examine the functions f for local maxima, local minima and saddle at $(0,0)$.

9. Examine the following functions for local maxima, local minima and saddle points.

   (a) $f(x,y) = x^2 - y^2$

   (b) $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy$

   (c) $f(x,y) = x^2 - 2xy^2$

10. Find the points of absolute maximum and absolute minimum of the function $f(x,y) = x^2 + y^2 - 2x + 2$ on the region $\{(x,y) : x^2 + y^2 \le 4 \text{ with } y \ge 0\}$.

11. $D = \{(x,y) \in \mathbb{R}^2 : x \ge 0 \text{ and } y \ge 0\}$ and $f(x,y) = (x^2+y^2)e^{-(x+y)}$. Show that

    (a) f is bounded on D.

    (b) f achieves its (absolute) maxima at a point of boundary of D.

    (c) $e^{x+y-2} \ge \frac{x^2+y^2}{4}$ for all $(x,y) \in D$.

12. Let $f(x,y) = x^2 + y^3$ and $g(x,y) = x^4 + y^6 - 2$.

    (a) Find set of points satisfying Lagrange's system of equations $\nabla f = \lambda \nabla g$ and $g = 0$.

    (b) Show that f achieves its maximum and minimum on the set $\{(x,y) : g(x,y) = 0\}$.

    (c) Find the points of maxima and minima of the function f subject to $g(x,y) = 0$.

13. Let $f(x,y,z) = xyz$ and $g(x,y,z) = x^2 + y^2 + z^2 - 12$.

    (a) Find the set of points satisfying the equations $\nabla f = \lambda \nabla g$ and $g = 0$ for $\lambda = 0$ and $\lambda \neq 0$.

    (b) Find the maximum and minimum values of f subject to the constraint $g = 0$.

14. Find the nearest point on the plane $x + 2y + 3z = 6$ from the point $(1,0,0)$.

15. Find a point on the surface $z = xy + 1$ which is nearest to $(0,0,0)$.

16. Find the extremum values of the function $f(x,y) = xy$ on the ellipse $\frac{x^2}{2} + \frac{y^2}{8} = 1$.

17. Using the method of Langrage's multipliers, find the three real numbers such that sum of the number is 12 and sum of their squares is as small as possible.

18. (a) For $n \ge 2$, let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x_1, x_2, \ldots, x_n) = x_1^2 x_2^2 \ldots x_n^2$. Find the maximum value of f subject to $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - c^2 = 0$.

    (b) Using (a), prove the inequality $(a_1 a_2 \ldots a_n)^{\frac{1}{n}} \le \frac{a_1 + a_2 + \cdots + a_n}{n}$, for any positive real numbers $a_1, a_2, \ldots, a_n$.