Indian Institute of Technology Guwahati

MA 102 Mathematics-II
Tutorial Sheet-2

Directional Derivatives and Tangent Plane

Section-I: Tutorial Problems

1. Let $f(x,y) = \frac{1}{2}\Big(\big||x|-|y|\big| - |x| - |y|\Big)$. Is f continuous at $(0,0)$? Which directional derivatives of f exist at $(0,0)$? Is f differentiable at $(0,0)$?

2. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a map defined by $f(x,y) = \begin{cases} (x+y)\log(x^2+y^2) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ Show that f is continuous at $(0,0)$. Find all possible directions $\mathbf{v}$ in which the directional derivative $D_{\mathbf{v}}f(0,0)$ exist. Is f differentiable at $(0,0)$?

3. Find the point on the surface $f(x,y,z) = 3x^2 - y^2 - z = 0$ at which the tangent plane is parallel to the plane $6x + 4y - z = 5$.

4. Find the equation of the surface generated by the normals to the surface $f(x,y,z) = x + 2yz + xyz^2 = 0$ at all points on the z-axis.

Section-II: Practice Problems

1. Find all $\mathbf{v} \in \mathbb{R}^2$ for which the directional derivative $f'_{\mathbf{v}}(0,0)$ exists, where for all $(x,y) \in \mathbb{R}^2$,
   (a) $f(x,y) = \sqrt{|x^2 - y^2|}$.
   (b) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   (c) $f(x,y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$

2. Let $f(x,y) = x^2e^y + \cos(xy)$. Find the directional derivative of f at $(1,2)$ in the direction of vector $(3,4)$.

3. The directional derivatives of a differentiable function $f:\mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$ in the directions of $(1,2)$ and $(2,1)$ are 1 and 2 respectively. Find $f'(0,0)$.

4. Let $u = (1,0,0)$, $v = \frac{1}{\sqrt{2}}(1,1,0)$ and $w = \frac{1}{\sqrt{3}}(1,1,1)$. Let $f:\mathbb{R}^3 \to \mathbb{R}$ be differentiable at $\mathbf{0}$. If $D_u(\mathbf{0}) = 1$, $D_v(\mathbf{0}) = 2$ and $D_w(\mathbf{0}) = -1$. Then find $f'(\mathbf{0})$.

5. Suppose $f:\mathbb{R}^2 \to \mathbb{R}$ is a function given by

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Let $\mathbf{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$. Find $D_{\mathbf{v}}f(0,0)$ of f and show that $D_{\mathbf{v}}f(0,0) \neq (\nabla f.\mathbf{v})(0,0)$.

(b) Show that the function f is not differentiable at $(0,0)$.

6. Let $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$ and $(u,v) \in \mathbb{R}^2$ be a unit vector. Show that directional derivative of f at $(0,0)$ in the direction of (u,v) exists if and only if $(u,v) = (1,0)$ or $(0,1)$.

7. For $X \in \mathbb{R}^3$, define $f(X) = \|X\|$. Let $X_o = (x_o, y_0, z_o) \in \mathbb{R}^3$ and $\|X_o\| = 1$,

   (a) Show that $\nabla f(X_o) = X_o$.

   (b) Find a unit normal to the sphere $f(x,y,z) = 1$ at X_o.

   (c) Find the equation of the tangent plane of the sphere $f(x,y,z) = 1$ at X_o.

8. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and $g(t) = (x(t), y(t), z(t)),\ t \in \mathbb{R}$, be a differentiable curve. Suppose that $f(g(t))$ attains its minimum at some point t_o. Show that $\nabla f(g(t_o))$ is perpendicular to $g'(t_o)$.

9. Let $f(x,y) = 6 - x^2 - 4y^2$. Find a vector which is perpendicular to

   (a) the curve $f(x,y) = 1$ at the point $(1,1)$.

   (b) the surface $z = f(x,y)$ at the point $(1,1,1)$.

10. Consider the cone $z^2 = x^2 + y^2$.

    (a) Find the equation of the tangent plane to the cone at $(1,1,\sqrt{2})$.

    (b) Find an equation for the normal line to the cone at this point.

11. Consider the surface $z = f(x,y) = x^2 - 2xy + 2y$. Find a point on the surface at which the surface has a horizontal tangent plane.